THE PHILOSOPHICAL FORUM
Volume XX, No. 3, Spring, 1989

REFLECTIONS ON THE LEGACY OF KURT GÖDEL: MATHEMATICS, SKEPTICISM, POSTMODERNISM

JOHN KADVANY

> Skepticism is an ability to place in antithesis, in any manner whatever, appearances and judgments . . . We call it an "ability" not in any subtle sense, but simply in respect of its "being able" . . . In the definition of the Skeptic system there is also implicitly included that of the Pyrrhonean philosopher: he is the man who participates in this "ability."
>
> Sextus Empiricus, *Outlines of Pyrrhonism*

Among mathematicians and non-mathematicians alike, Kurt Gödel is best known for a pair of remarkable theorems he proved in 1931, his so-called First and Second Incompleteness Theorems. Both of these theorems apply to axiomatic systems intended to represent the corpus of mathematical knowledge as being generated from a finite or, more generally, recursive set of axioms by the application of a finite set of precisely specified rules of inference. Russell and Whitehead's *Principia Mathematica* is perhaps the most famous such system, though today a good student of mathematical logic can enumerate a half-dozen candidates for mathematical foundations, all having their own exotic but mutually inconsistent consequences for various mathematical conjectures. It is possible, for example, for texts in mathematical logic to pass through successive chapters, each one on a distinctive set-theoretic framework for the whole of mathematics, without the author having as much as to pause to justify such a mind-boggling theoretical stance. The advanced student of foundations must be equally at home in Gödel's constructible universe of sets, the so-called "playful" universe based on the formal treatment of a game-theoretic interpretation of quantifiers, and in models of mathematics with a cornucopia of large cardinal axioms regarding the range of allowable infinities.

It is simply a basic feature of contemporary foundations of mathematics that from the theoretical perspective provided by mathematical logic, mathematics as a whole is as fragmented and bifurcated a mode of intellectual activity as one might find in the imagination of the most redoubtable postmodernist. I use this term only descriptively, and wish to emphasize the difference between mathematical practice and metamathematical theory—though even the former is far from providing the kind of unified program that can stand as a potent antidote to postmodernism's undesirable normative aspirations. This relatively novel state of logical theory is by no means all that problematic or catastrophic for mathematicians or logicians, and the mere use of a new epithet neither aggrandizes the word nor revolutionizes our understanding of mathematical practice. But there are some philosophical lessons to be found in pursuing this characterization of metamathematical thought, not the least of which is in demonstrating the basic weakness of postmodern epistemology, namely its chronic and self-deluding ahistoricism.

What then is it about Gödel's inventions that made them a primal source for a recognizably progressed postmodern condition in contemporary mathematical philosophy? This remarkable explosion of incommensurable theories could not have become so second nature without Gödel's metamathematical techniques, so it seems reasonable to conjecture that the current view of mathematics as a congeries of mutually incompatible systems, none of which is necessarily preferable to another as *the* representation of mathematical thought, owes a good deal of its power to some basic epistemic features of the proofs and content of Gödel's results. Indeed it was Gödel who transformed a whole arena of debate straddling the boundary between philosophy and mathematics into a new field of mathematical thought,[1] and it is this transformation I wish to approach via the epistemological problem of the varieties of mathematical foundations.

The simplest observation of how Gödel's Theorems create a postmodern condition begins with the First Incompleteness Theorem. This theorem says, in effect, that a consistent axiomatic system strong enough to prove some weak theorems from elementary number theory, requiring only the operations of addition and multiplication, but not either operation separately,[2] will be *incomplete:* there will always be mathematical sentences formulated in the syntax of the system under consideration that are neither provable nor refutable in the system, and these sentences are said to be *undecidable* with respect to the system. Since an undecidable proposition and its negation are each separately consistent with the base system, one can extend the old system to two mutually incompatible new ones by adding on the undecidable sentence or its negation as a new axiom. The classical example of this procedure is the generation of non-Euclidean geometries by adding the negation of the parallel postulate to the axioms of elementary geometry without the parallel postulate. The new systems so constructed also have new undecidable sentences, different from the originals,

and the process of constructing new undecidable sentences and then new systems incorporating them or their negations goes on *ad infinitum,* like a branching tree which never ends. One can indeed formalize this argument to prove that there is a continuum of mutually inconsistent theories all containing Peano arithmetic.

Gödel's First Theorem showed that undecidability was endemic to mathematical reasoning and a major portion of foundational resesarch since the 1930s has been involved in exploring *mathematically* important conjectures[3] that are undecidable with respect to various "foundational" systems—the ironic quotation marks now being mathematically necessary. Gödel's original undecidable sentence is of no known mathematical interest apart from its undecidability in elementary arithmetic and its primary role in logical research, and for many years it was an open problem to find a non-metamathematical sentence similarly undecidable in Peano arithmetic; the first such sentence was found in 1977 by Jeff Paris and Leo Harrington.[4] Another line of research following from the model set by the First Theorem turns the undecidability issue around. Why not investigate alternative mathematical systems that *do* decide some of the mathematically interesting undecidable propositions one way or the other? Within any one of these systems, one can investigate those new theorems which are provable or refutable, but undecidable with respect to relatively lean systems such as Zermelo-Fraenkel (ZF) set theory without the axiom of choice. Cantor's Continuum Hypothesis, regarding the order of infinity of the real number line, is the most famous undecidable statement investigated in this way, but there are today several such propositions of mathematical, and not only metamathematical, interest. One of the dominant strategies therefore in post-Gödelian foundational studies is reflected in these attempts to prove to mathematicians that metamathematics has a direct bearing on concrete mathematical problems,[5] but this progress has been made mostly *across* several foundational theories, primarily ones with specialized consequences for real analysis and properties of the real number line.

Once it was seen that one could profit from undecidability by leap-frogging into new systems in which the undecidable sentence was provable or refutable—and not *merely* this, for the point is to decide as many important sentences as possible from a minimum of additional postulates—it is not difficult to see with hindsight how mathematical invention and creativity would lead to the current state of the discipline. It would have taken almost cynical foresight to expect that none of several promising avenues of research would prove to be a satisfactory post-incompleteness "foundation," and that the resulting expression of mathematical practice was therefore an open field whose boundaries would be difficult to demarcate in terms of objective progress. But this answer, sketchy as it may be, makes the self-divided condition of meatamathematics a somewhat superficial aspect of constructions made possible by undecidability; it provides little in the way of an epistemological context. The argument from undecidability, joined with Gödel's own heuristic advice to mathematize metamathematics, seems basically right, but it also seems adventitious. My own preference is to eschew coincidence in the history of ideas, so when we notice among mathematicians the rapid cultivation of a kind of theoretical activity associated with postmodernism, even though many might label such a general call to anarchy a kind of comedy of the higher lunacy, it behooves us to inquire further into the nature of so striking a similarity.

II

Classical Pyrrhonian skepticism provides an explanation of the analogies between truly anarchistic theories of knowledge and the consequences of Gödel's results. Gödel's work has an elementary skeptical core which is conveniently elaborated via the vocabulary of "standards" and "criteria" of Pyrrhonian skepticism. Both Incompleteness Theorems settle specific problems about criteria of mathematical truth by defining the extent to which a standard of mathematical *proof* can satisfy conditions for a criterion of mathematical *truth*. Are the rules of proof complete, in that they prove or refute all well-formed sentences in the syntax of the formalism?; and are the rules provably self-consistent within the formalism?—these are the two questions of criterial adequacy on mathematical truth answered by Gödel in the negative for a broad class of axiomatic systems.

The content of the Second Theorem may also be placed in Pyrrhonism. The Second Theorem states, in crude, that the *consistency* of the system under consideration, say *Principia Mathematica,* is itself not provable in the system, unless the system is inconsistent, in which case all sentences are provable. If the formalism is thought to be an adequate representation of mathematics as a whole, this means that we lack an internal, mathematical justification against the possibility of proving "1 = 2", or absolutely any sentence formulated in the syntax of, say, *Principia Mathematica*. *Mathematics has no mathematical foundations* that can be justified without a vicious mathematical circle, or an infinite regress of criteria of mathematical truth. Stated in a positive spirit, the consistency of mathematics is a conjecture which may be falsified by the discovery of a proof of inconsistency. But the basic antifoundational consequence of the Second Theorem is an instance of the skeptical *problem of the criterion,* as formulated by Sextus Empiricus in his *Outlines of Pyrrhonism:*

> . . . in order to decide the dispute which has arisen about the criterion, we must possess an accepted criterion by which we shall be able to judge the dispute; and in order to possess an accepted criterion, the dispute about the criterion must first be decided. And when the argument thus reduces itself to a form of circular reasoning the discovery of the criterion becomes impracticable, since we do not allow them [the Dogmatic philosophers] to adopt a criterion by assumption, while if they offer to judge the criterion by a criterion we force them to a regress *ad infinitum*.[6]

The skeptic here asserts what Gödel shows can be established mathematically when consistency is taken as the issue of dispute between mathematical dogmatists and mathematical skeptics: either the consistency of mathematics must be assumed outright or invoked by some extramathematical standard, or we are forced into an infinite regress of stronger and stronger foundational theories. The antifoundational content of Gödel's Second Incompleteness Theorem may be taken to show that Gödel proved that the ancient skeptical problem of the criterion can be explicitly formulated as the conclusion of a mathematical proof. The problem of the criterion is generally used to attack theories of knowledge aiming to show that there is *some* certain knowledge, some knowledge that is infallibily known to be true, as was thought of mathematics since Aristotle. Hilbert, above all, restricted mathematical dogmatism to a clearly defined desideratum: *within* mathematics, *only* a certain set of sentences is provable and *absolutely* no more. Gödel's Second Theorem implies that the consistency of *Principia* can be mathematically proven only by conjecturally assuming the consistency of *Principia* outright (which is what mathematicians implicitly do in practice), or by reducing the consistency of *Principia* to that of a stronger system, thereby beginning an infinite regress. The two alternatives, of dogmatism or an infinite regress, form the conclusion to the problem of the criterion, and the problem may be viewed as a generalization of Gödel's Second Theorem.

The skeptic generally develops the problem of the criterion via a series of mutually exclusive and exhaustive choices for "locating" the justification for a criterion of truth, with the aim of showing that the problem of foundations, rather than being eliminated, has been moved from one place to another. To invoke a Kantian standard, for example, that the truths of mathematics are guaranteed by the transcendental structure of our sensible intuitions of space and time, is, for the skeptic, to introduce a *second* standard which now guarantees the truth of the first, and the skeptical reply is to ask again whether this standard is self-validating, or whether it too depends on some further criterion of truth. Gödel showed that as long as we remain within a broad class of *mathematically* definable criteria, the alternatives posed by the Pyrrhonist for the foundations of mathematics are correct. Whether or not one thinks the problem of the criterion is valid in general is here irrelevant, since Gödel showed with great precision that for his proof-systems, their internal consistency could be non-dogmatically proven only by appealing to an alternative standard of truth whose strength, relative to the first, could be determined in mathematically exact terms. As long as one considers only mathematical criteria of truth, as opposed to, for example, Kant's transcendental theory of consciousness, or the set-theoretic Platonism Gödel developed in the 1940s, or the neo-Kantian psychologism of intuitionism, Gödel's Second Theorem proves that there is a rigorous interpretation of the problem of the criterion when the standard of mathematical proof is used as a criterion of mathematical truth. And like the problem of the criterion, Gödel's proofs have complete generality: they may be applied to all kinds of mathematical systems with the same devastating consequences.[7]

So far this mostly amounts to a restatement of the standard interpretation of the Second Theorem, but the reformulation points to another aspect of Gödel's work describable through Pyrrhonism that reappears in postmodernism. While the Second Theorem may be taken as expressing an instance of the problem of the criterion, the undecidability conclusion of the First Theorem can be described as an instance of skeptical *isostheneia*[8], or the relative equipollence of arguments for two inconsistent characterizations of the real or that which is the case.[9] For us this means provisionally taking the *proofs* of Peano Arithmetic as the *truths* of arithmetic, or the proofs of *Principia*, ZF, ZFC, ZF + AD, etc., as the truths of a universe of sets—"The Real World," as set theorists often say. As long as the base system is consistent, both the undecidable sentence and its negation are consistent with the system, and thereby provide two versions of "the real," which is inconsistent with the proof-system considered as a standard of truth. This establishes an instance of Pyrrhonean *isostheneia*. As with the problem of the criterion, Gödel showed that the skeptical technique of constructing *isostheneia* is rigorously possible within mathematics when certain formal standards of proof are put forward as informal standards of mathematical truth. And the skeptical trope of *isostheneia,* like the problem of the criterion, is not to be taken as necessarily coming from or being directed against a specific philosophical or mathematical position. Gödel's argument is a readily (informally) generalizable scheme adaptable to mathematical systems based on numbers, sets, or computational models of many kinds, and just as Sextus distinguishes the general features of skepticism from its "special" part, in which skeptics argue against "each part of so-called philosophy,"[10] Gödel's work too is ultimately at home in specific applications, as opposed to a purely mathematical theory of undecidability.

It is important that the skeptic, in developing *isostheneia* and the problem of the criterion, does not thereby *reject* the proposed standard as a standard of knowledge, but only curtails its claim to absolute truth: "We do not employ the arguments against the criterion by way of abolishing it but with the object of showing that the existence of a criterion is not altogether to be trusted, equal grounds being presented for the opposite view."[11] Undecidability for the skeptic, as for the logician, does not establish that there is absolutely no criterion of truth, but only that "nothing is absolutely a criterion of truth."[12] While Gödel's undecidable sentence is *true*—for it asserts of itself that it is unprovable, and

that is the case (in the standard model)—*that* expression of truth is unprovable within the original proof-system to which Gödel's Theorems apply. This holding back of Pyrrhonian criticism is closely related to the precise form of the Incompleteness results. Both Incompleteness Theorems, in asserting the unprovability of mathematical propositions, crucially depend upon the consistency of the system under consideration. If the system is inconsistent, then all propositions are provable in it, so there are no undecidable sentences, and one can "prove" consistency in particular. This triviality is central to the entire structure of Gödel's Theorems, since Gödel proved that *if* arithmetic (or *Principia,* ZF, etc.) is consistent, *then* consistency cannot be internally proven and there are undecidable sentences with respect to the axioms at hand. Gödel's Theorems are *conditional* theorems whose antecedents include the statement of consistency of the axiom system under view. Formally, Gödel's Theorems are Δ_2^0 sentences in the arithmetic hierarchy, both being theorems of the form $P \rightarrow Q$, with P and Q both Π_1^0 or universal sentences. Epistemologically, the conditional form of Gödel's theorems makes them part of a critique directed against a potential foundation for mathematics. We know metamathematically that the conditional form cannot be reduced to a lower level of the arithmetic hierarchy, so any instance of Gödel's proofs is always conditionalized by the consistency statement for the "candidate" for a mathematical foundation.

This technical proviso is neatly echoed in Pyrrhonian methodology. When the skeptic is questioned about the status of his own skeptical pronouncements, just as Gödel might be questioned about the status of his (incorrectly stated) *un*conditionalized Theorems, the skeptic replies that his conclusions are framed only within the context of a dialectical debate between himself and an antagonist who claims certainty based upon some given criterion of truth. The skeptic does not claim to establish *isostheneia in opposition to,* e.g., a Stoic criterion of truth, but only to be able to develop *isostheneia beginning immanently from Stoic assumptions,* and thereby show that the Stoic's standard leads him to abandon his account of the real. Skeptical arguments are propounded in a series of stages, and the truth of the dogmatic starting point is only conditionally assumed in order to carry to completion the construction of isostheneia; the formal residue of Gödel's immanent critique[13] of the Hilbert program is found in the conditional antecedents of consistency. The Pyrrhonian does not deny knowledge, just as Gödel does not deny consistency or completeness *simpliciter,* but only by and because of a prior "assertion" based in an opposing system. Sextus says that "Arcesilaus did not, in principle, establish any criterion; but those who think he did establish one ascribed it to an attack,"[14] and similarly Gödel asserts the unprovability of consistency and undecidability from within, meaning that just as the Skeptic borrows vocabulary and logic from the Stoa, Gödel constructs a proof-predicate specific to the system at hand. There is no opposition of principles but a single immanent critique from within one system, and an absence of any genuine positive doctrine. Montaigne remarks in his skeptical masterpiece *The Defense of Raimond Sebond* that Pyrrhonists need a "negative language" in which to express their criticisms without overstating them, and restricted to metamathematics this is one of Gödel's great achievements: the construction of a complete theoretical apparatus for formulating and proving relative undecidability and unprovability conditions. Undecidability, like *isostheneia,* is always based on a particular "concession" of dogmatism, and thereby only conditionally refutes an absolute criterion of truth, which is not to show that there is absolutely no criterion of (mathematical) truth.

It is tempting to try to obviate the conditional dependence of Gödel's Theorems on the consistency of arithmetic by a kind of metalogical argument: Gödel proved that he could show conditional undecidability if arithmetic is consistent, and if arithmetic is inconsistent, then everything is provable, including Gödel's Theorems. So at least one could say absolutely that either arithmetic is consistent or it is inconsistent, and in either case Gödel's Theorems are provable. The point of this metalogical rejoinder would be to mitigate Gödel's antifoundational conclusions, the argument from metalogical *tertium non datur* being taken to show that Gödel's conclusions are only as strong as the systems in and against which they are constructed, and that his antifoundationalism is therefore self-refuting or self-annulling. Do we not at least know, mathematically, the conditional assertion of the unprovability of consistency, even as that assertion means that *everything* may be provable anyway?

This attempt at weakening the impact of Gödel's antifoundationalism accepts that while, as Hume says, "the understanding when it acts alone, and according to its most general principles, entirely subverts itself, and leaves not the lowest degree of evidence in any proposition, either in philosophy or common life,"[15] such subversion itself depends on rational principles to exist as an argument at all. Hume's own reply to this counter-skeptical rejoinder acknowledges that because skeptical arguments are constructed immanently, skeptical reasoning is at least as strong as that to which it is opposed. But at the same time, the inevitability and generalizability of skeptical conclusions still does *not* provide a positive basis for rationality of the kind envisioned by a dogmatist or foundationalist. If the truth of the dogmatic starting point is taken to be thrown into doubt by skeptical argumentation, then such doubt extends, perforce, to the skeptic, and completes the annihilation of reason from within:

> If the sceptical reasonings be strong, say they, "tis proof, that reason may have some force and authority: if weak, they can never be sufficient to invalidate all the conclusions of our understanding." This argument is not just; because the sceptical reasonings, were it possible for them to exist, and were they not destroyed by their subtlety, wou'd be successively both strong and weak, according to the successive dispositions of the mind.[16]

For Hume, the skeptical arguments are simply as strong as any rational argument, *and* self-annulling. The exercise of reason to ground itself produces an inevitable cycle in which reason subverts itself into skeptical conclusions, which, if in turn are attacked as being *only* as strong as the reason to which they apply, do not then weakly justify rationality, but instead consume it and skepticism as well. The metalogical argument which attempts to mitigate Gödel's antifoundationalism is the last step before a parallel annihilation in mathematics, because the metalogical alternative that in inconsistent mathematics one can prove anything at all is, in this context, incoherent. It represents a Humean condition of reason destroyed in trying to repel a skeptical assault, shown by "proofs" that prove anything and everything, i.e. a meaningless, undifferentiated enumeration of symbols.

The notion that skeptical arguments consume themselves along with the knowledge against which they are deployed is not new with Hume. It can be found in ancient Pyrrhonism, where the final stage of skeptical dialectic was often consummated in the psychological state of *ataraxia,* or unburdened tranquility, and the epistemological state associated with the *epoche,* or suspension of belief in absolute truth. But what is new in Hume and in modern philosophy is a persistence in taking skepticism to its logical extreme while simultaneously denying the practical possibility of living out a skeptical life in which one really doubts or suspends judgment upon all questions of truth. The pragmatic question of living out skepticism was an important issue after the rediscovery of Greek Pyrrhonist writings at the end of the Renaissance. Pascal in his *Pensees,* after saying that "*le pyrrhonism est le vrai*"[17] goes on to ask what it may be like to be a complete Pyrrhonist: does one doubt whether one is awake, or is being pinched, or is being burned, or whether he exists, or even doubts? "We cannot go so far as that," writes Pascal, "and I lay it down as a fact there never has been a real complete sceptic. Nature sustains our feeble reason, and prevents it raving to this extent . . ." It was for this inability to live out the reality of Pyrrhonian arguments that Antoine Arnauld also remarked that "Pyrrhonism is not a sect of people who are persuaded of what they say, but . . . is a sect of liars." And so too Hume, who likewise remarks that "the great subverter of *Pyrrhonism* or the excessive principles of scepticism, is action, and employment, and the occupation of common life." But this does not then throw skepticism into doubt for Hume. It means that those who choose to inquire into the philosophical problems leading to skepticism are necessarily led to lead a double life: they hold on the one hand a theoretically correct rationalism-leading-to-skepticism, and on the other a naturalistically conditioned dogmatism, equally correct in a pragmatism needed in order to carry on with the business of life. This, roughly speaking, is what the historian Richard Popkin calls Hume's "schizophrenic" skepticism.[18] "No philosophical dogmatist denies," writes Hume in his *Dialogues Concerning Natural Religion,* that there are difficulties both with regard to the senses and to all science: and that these difficulties are in a regular, logical method, absolutely insolveable. No sceptic denies, that we lie under an absolute necessity, notwithstanding these difficulties, of thinking, and believing, and reasoning with regard to all kind of subjects, and even of frequently assenting with confidence and security. The only difference, then, between these sects, if they merit that name, is, that the sceptic, from habit, caprice, or inclination, insists most on the difficulties; the dogmatist, for like reasons, on the necessity.[19]

If I am correct in outlining the skeptical content of Gödel's work, then the conditions set for mathematical reasoning, implied by Gödel's Theorems, are much the same—and that because the "practical" and theoretical modes of mathematical activity coincide, the contemporary mathematician, insofar as he is represented by metamathematical research, is the embodiment of Hume's schizophrenic skeptic. Practically, the working mathematician has to build his work on some provisional foundation, setting bounds on the kinds of mathematical entities he will countenance as legitimate, and on the inferences allowable over this domain. We know from Gödel's work that the innocuous dogmatism involved in such assumptions is mathematically necessary, and the mathematician is forced into such dogmatism by the practical need to carry out research. At the same time, he knows very well that nearly any set of assumptions he chooses are subject to Gödel's skeptical conclusions, so at least as far as *mathematical* foundations are concerned, he does live out Hume's portrait of the skeptic with the split personality. And while the habit of cultivating such theoretically self-destructive intellectual exercises is also characteristic of postmodernism, their startling but comprehensible presence in mathematics shows that this "new" movement is not as radical as either some of its critics or proponents would have us believe. At least it should be clear that a pure postmodern conditions is not so extraordinary: it is a mirror of something like the proliferation of alternative versions of mathematics by contemporary logicians, which in turn has an elementary philosophical basis, being a mathematical expression of the tropes of ancient skepticism.

III

While it is normatively unattractive to identify, if only in part, a half-century of distinguished mathematical research with a kind of logic of cultural and epistemic disintegration, it is best to confront the consesquences of this identification squarely, and not facilely dismiss it as mere analogy, or just another philosophical peg on which to hang concrete results. So let us continue the insight that Gödel has formalized several core techniques of Pyrrhonism into a closed interpretive system, one made up of an initial informal mathematical argument, which is then projected by a thought experiment (and it is irrelevant whether these formalizations are actually carried out) into the formalism the argument takes as its object of study, and, in which, theorems, meta-theorems, and meta-meta-theorems on relative undecidability and the relative unprovability of consistency are quite coherently formulated and proved.

Gödel, from this perspective, would have achieved the perfection of Pyrrhonism in mathematics, apparently avoiding the critical dangers of the *peritrope*, or reversal, often deployed against the skeptic to attempt to show the failure of *his* expression of knowledge to meet an implied or expressly stated criterion of truth. The reversal turned back against the skeptic is justified on skeptical grounds because the skeptical notion of appearance or *phainomena* is not referential (e.g. as to sense-appearance, as is sometimes anachronistically thought), but serves to identify the skeptic's *topic-neutral* withdrawal from epistemic claims of how the world truly is. In particular, the skeptic's own philosophical discourse is included under appearance as the expression of the criterion of appearance by which the skeptic says he leads his own life of perpetual inquiry.[20]

What is characteristic of the peritrope is not just its conceptual turnabout, but the way in which the peritrope points back to previous steps or stages or argumentation, and includes the performance of these stages within the scope of the conceptual reversal. In ancient times the concrete circumstances in which the peritrope was applied typically would be an oral debate between two opposed speakers:[21] not a unified abstract argument, but a sequentially articulated series of moves and countermoves.[22] The invocation of the peritrope brings about the conceptual self-application of an argument as it points to the specific delivery of the argument in speech, so that the peritrope functioned as the oral equivalent of critically deployed, ironizing quotations. It is the *realization* of self-application that gives an instance of the peritrope its point, not just the reversal *in abstracto*, as the skeptic empirically identifies the appearance of a criterial standard in the speech of his opponent. The peritrope is then uncoincidentally at the heart of the problem, perceived so accurately by Pascal and Hume, of how the skeptic lives out a skeptical life, since the peritrope provides the internal starting point for a critique of skeptical practice; but as mentioned above, at least Hume felt that such a critique was not entirely possible since it led to a disintegration of reason instead of reason's rescue. But the peritrope has *positive* uses as well, meaning that the "turning of the tables" effected by the reversing gambit may show that, *in the actual execution of the argument*, the speaker of the argument *does* indeed meet the criterion of truth (viz. appearance) that he uses as his own. The *peritrope* appears not in the *content* of either of Gödel's Theorems, as do *isostheneia* and the problem of the criterion, but *as the proof of the Second Theorem*. Indeed it is because of this unusual proof that Gödel's Second Incompleteness Theorem is one of the most remarkable *corollaries* in the whole of mathematics.

The Second Theorem is proved by a thought-experiment in which one considers the arguments that have gone into the machinery (the Chinese Remainder Theorem, the elimination of inductive definitions, etc.) making the First Theorem possible, and then sees that these informal mathematical arguments are all entirely formalizable within the very system under consideration; the final step is to realize that the only assumption needed to carry out the proof of the First Theorem in addition to the arithmetical or set-theoretical axioms *is the formalized statement of consistency*. A weak version of the First Theorem assumes consistency externally, appealing, for example, to the existence of a standard model of arithmetic, perhaps to establish the truth-value of the undecidable sentence; but the strong version appeals to the thought-experiment, and, by a careful inspection of the First Theorem's *proof, improves* the theorem to require only the formal statement of consistency (illustrating what Imre Lakatos calls, in *Proofs and Refutations,* an instance of *lemma-incorporation* in the informal proof). By the thought-experiment, it is shown that the First Theorem is provable in, say, *Principia,* and from this fact, with a few additional lines of elementary argument, the Second Theorem follows—i.e. that consistency implies that consistency is unprovable. The essence of the proof of the Second Theorem is that the proof formalizes the informal mathematical proof of the First Theorem.

There is no self-refutation here, as is typical with applications of the peritrope, but there is an *existential reversal* appearing at a precise stage of a sequentially elaborated argument. Insofar as the invocation of the peritrope proceeds by applying part of the criterion of truth at issue to a meta-criterial claim about it, the proof of the Second Theorem falls right under that classification, the criterion at issue being the formalization of a certain fragment of *in*formal mathematical discourse. The First Theorem is itself not a formal object, it is an informal mathematical proof through which a claim is established about the limitations of a certain formal model of mathematical reasoning. As this happens, *two versions* of the First Theorem come into play: the informal one proved by Gödel, and the formal representation of *that* proof, a representation immediately needed in the proof of the Second Theorem. When, hypothetically, the tables are turned on Gödel and it is ''proposed'' that the standard of truth which is applied to his object of study become the standard for the very propositions he is making—that is, when Gödel's First Theorem is *translated* from the informal domain of mathematical argumentation into the formal domain—this peritrope, far from contradicting some feature of the First Theorem, turns the First Theorem into the Second. In a famous passage of the *Theaetetus* where the peritrope is being used against Protagoras' relativistic ''man is the measure'' doctrine, Plato describes this particular self-refutation as ''the most exquisite argument.'' Some two thousand years later, in his *Set Theory and the Continuum Hypothesis,* Paul Cohen, who proved the undecidability of Cantor's Continuum Hypothesis for a broad class of axioms for set theory in 1963, remarked of the proof of Gödel's Second Theorem that it was the most subtle point in the development of mathematical

logic.[23] I suggest that Cohen and Plato are reacting to precisely the same argument deployed centuries apart with equal amazement. Only unlike Protagoras (at least according to Plato), Gödel is able to dwell within his mathematical skepticism without the pain of self-refutation, since his constructive and generalizable method for the exposure of mathematical *isostheneia,* and his mathematical formulation of the antifoundationalism of the problem of the criterion also allows for the iterated self-application of concepts in a formal syntactic structure consistent with the skepticism which it follows as a *coup de grâce*.

The ingenious peritropical technique by which the Second Theorem is derived from the First does not indeed preserve *all* the properties one might wish a metamathematical proof to have. What one would hope for initially, and probably expect, is that the Second Theorem would not essentially depend on the form the translation takes. For example, there are many alternative Gödel-numberings possible and it should be clear that a choice among these does not affect the outcome of the proof. *Modulo* these details, such a proof would be *invariant* with respect to the choice of translation. This invariance would provide a kind of closure on Gödel's results, which would then be distinguished by their rigorous conceptual completeness.

But the Second Theorem does not have this property, and for mathematically important reasons. When Gödel published his epochal 1931 paper, *On Formally Undecidable Propositions of* Principia Mathematica *and Related Systems I,* he intended to elaborate details of his proofs in a later work. This work was never published, apparently due to the immediate acceptance of Gödel's proofs among mathematicians. It is likely that this sequel would have contained a study of the exact form of the formalized consistency statement, the important antecedent that needs to be made part of the hypotheses of Gödel's results. In the published paper, the explicit form of the consistency statement is relegated to a footnote, inauspiciously introducing into mathematics one of the most curious logical entities ever discovered. The form given by Gödel is correct, but it is only one reasonable formalization of consistency. While the formalization given by Gödel is precisely the one, up to provable equivalence, needed to make the proof of the Second Theorem work, there are alternative formalizations of consistency which make the Second Theorem false; i.e. for these choices, consistency *is* internally provable. Admittedly, these constructions are unintended counterexamples to the Second Theorem—what Lakatos calls ''monsters''—and play the valuable role of forcing a more painstaking analysis of the Incompleteness proofs to determine the further conditions, left unspecified by Gödel, which show that the choice of canonical representatives for provability and consistency is theoretically jusitifiable. The core of this theory of formalized proof predicates and consistency statements is a set of conditions examined originally in the context of Gödel's work by David Hilbert and Paul Bernays in the second volume of their *Grundlagen der Mathematik,* published a few years (1939) after Gödel's paper, though other publications suggest that the idea behind the proof-conditions they identified as sufficient for the Second Theorem was in the air somewhat earlier. While it may originally have been thought that the Hilbert-Bernays conditions, later improved by Löb, merely identified some "obvious" properties that a formalized proof-predicate (and, by implication, a formal consistency statement) should have, the history of Gödel's results from the 1930s on shows, as Lakatos again would say, that Gödel *did not prove what he set out to prove* in his Second Theorem. Not that there is a mistake in the Second Theorem, but that as in so many cases of the historical development of a mathematical proof, the elaboration of lemmas and conditions needed to make Gödel's proof work eventually led to a novel and unexpected theoretical treatment of the concepts involved, concepts which could not have been anticipated at the inception of the original result. It is precisely this necessary entry of mathematics into its own history that delivers Gödel from postmodernism, if not entirely from mathematical skepticism.

Already in 1936 it was discovered by Barkley Rosser that desirable technical improvements (the elimination of ω-consistency in favor of simple consistency) in the First Theorem apparently required a special choice for the formalized proof-predicate. But this would only be the first time that the problem of the "correct" translation-procedure into formal metamathematics would be raised, and always motivated by technical and theoretical issues in Gödel's proofs. Rosser created his formal proof-predicate *Prov* from "x is a proof of y and no shorter proof of not-y exists," instead of Gödel's *intensionally correct* "x is a proof of y," denoted by *Bew,* for *beweis*. These two formulas are *extensionally* equivalent, i.e., they enumerate exactly the same sets of Gödel-numbers of proofs and theorems. While Rosser noted that the two proof-predicates have very different properties, he does not comment on building a consistency statement using his new proof-predicate, but indeed such an extensionally correct but intensionally incorrect formalization is provable and is a Lakatosian monster for Gödel's Second Theorem. The problem—you can barely call it a controversy—over the correct form of the consistency statement simmered in the mathematical literature for many years. There were criticisms that results related to the Second Theorem, and the intensional formulation of consistency, were being misstated, but there was also concern over unsolved theoretical difficulties directly caused by intensionality.[24] In fact one can still find misrepresentations of the Second Theorem in popular accounts of Gödel's work, and in histories of modern logic,[25] the mistake being the assertion that Gödel proved the impossibility of proving *any* form of consistency of a formal system within the system. This is just false. The translation problem arose again in 1952 when Leon Henkin posed the problem, later settled by Löb, of whether the self-referential mathematical sentence asserting of itself that it *is* provable, was in fact provable and therefore

true—unlike Gödel's undecidable sentence, which assets its own *un*provability, is not provable, and is therefore true. Henkin's problem, Georg Kreisel pointed out, was ill-formed as stated[26], since its resolution could depend on the specific formalization chosen for provability. Taken in isolation from its historical context, Kreisel's objection may seem a bit pedantic, since "everyone knows" the preferred choice of the provability predicate. But that kind of reaction betrays a historical forgetfulness of the fact that Gödel had taken logicians over a threshold from philosophy to mathematics and that there were still theoretical issues left unresolved by the *choices* Gödel had made. These developments mark the transition from Gödel's initial *naive conjecture*—that consistency is unprovable and this result does not essentially depend upon the formalization chosen—to a theoretical, and therefore mathematically acceptable treatment of consistency and proof-predicates.

It was not until the later 1950s that Solomon Feferman investigated in some detail the difference between the First and Second Theorems, and how the truth of the Second Theorem depends upon intensional choices for the consistency statement.[27] Feferman deals directly with the question of whether the "natural," intensionally correct definition of formal consistency, *Con,* can be theoretically justified as the sole choice, and he essentially answers this question in the negative. The Second Incompleteness Theorem, unlike the First, depends upon choices that cannot be completely justified within Gödel's original epistemological framework, which, I presume, does not entail intensionality. Feferman shows that there are *completely general conditions,* and not merely isolated counterexamples, under which consistency is provable using an extensionally correct, but intensionally deviant (from the retrospective standpoint of Gödel's results), representation of formal provability:

> Rather than contradicting Gödel's second underivability theorem [these results] show the importance of a precise method of dealing with consistency statements, at any rate for theories with infinitely many axioms [such as Peano Arithmetic] . . . A first reaction following such realizations might be to restrict attention to a certain class of "natural" formulas a in problems of arithmeticization . . . However, we shall obtain . . . results through the use of arbitrary formulas a which should be of interest even to those who would otherwise thus restrict attention. There is nothing "wrong" with the use of arbitary formulas a; rather, the guiding consideration should be to investigate how different restrictions on the *choice* of a affect the results by arithmeticization.[28]

Feferman uses an ordering on representations of axioms to provide a framework for choosing a single arithmetic sentence (up to equivalence) to express consistency, the "natural" choice being the minimum in this ordering, but for arbitrary (i.e. those with an infinite number of axioms) arithmetical systems, he shows that no such minimum (or maximum) in the ordering exists. "The moral of these theorems," writes Feferman, "is not to reject the use of particular numerations for known particular axiom systems *A;* for example, we still consider "natural" the definition . . . of *P*[eano Arithmetic] as a finite set of axioms and axiom schemata. Rather, it is to reject the use, as a well defined idea, of sentences *Con(A)* associated with arbitrary systems *A* . . . there is no natural or favored description . . . of an extensionally given infinite set of axioms," [29] and, under fairly unrestrictive conditions, there is also no natural choice for expressing consistency. What is natural is a matter of historical choice made against a background of mathematical tradition adapting to a radical new set of ideas. Another way of putting Feferman's point is this: if one performs the thought-experiment of carrying out the proof of Gödel's First Theorem using only *extensional* notions of provability—notions depending only upon the numbers formally representing provable sentences, but not depending on how these numbers are variously denoted—the pathological consistency statements will "appear" completely normal and the Second Theorem need not hold. It is only when one introduces the needed intensional conditions taken from additional *informal* mathematical arguments that Gödel's thought-experiment works *the way it should.*

After Feferman, the theoretical characterization of *Prov, Bew,* and *Con* took a different turn. There is now a well-developed theory based on the semantics of modal-logical systems that provides a good mathematical justification for the choice of *Bew* over *Prov,* and *Con* over a *Con** constructed from *Prov.* For example, using the modal representation of proof-conditions, it can be shown why the Rosser undecidable sentence cannot be replaced by one using the canonical proof-predicate, and how to distinguish *Bew* and *Prov* by critical properties, among the most important being that the coextensiveness of *Bew* and *Prov* is not provable in Peano arithmetic. These results provide a non-ad hoc theoretical differentiation between *Bew* and *Prov* that was not available to either Gödel or Rosser, and thereby explain the pathological status of *Con*.* It is results such as these which justify the important qualification of the unprovability of consistency, that, as George Boolos says in *The Unprovability of Consistency,* "although *Prov* is coextensive with *Bew, -Prov*("0 = 1") does not express the consistency of *PA; at best, it merely asserts* that every proof of "0 = 1" that there might be occurs later than some proof of "−0 = 1"." [30]

But what has been forgotten is that these characterizations are mathematically, rather than philosophically motivated, *and mathematically justified.* This not-completely benign historical neglect, combined with my prior observations of Gödel's implicit skepticism, provides a choice for the contemporary logician: either (a) do away with the arbitariness of a pure mathematical skepticism lead-

ing to postmodernism, by recognizing the specific historical development that led to our current understanding of incompleteness; or (b) neglect the history, and slip back into proliferating incommensurability. Option (a) does not eliminate mathematical skepticism, but it also shows that no simplistic epistemology is going to explain the complex interaction between philosophy and mathematics that took place in Gödel's work and its aftermath. It also shows that we do not deal with *any* kind of incommensurability in mathematics, but only ones devolving from special problem-situations. Option (b) is ahistoricism, which is not philosophically inconsistent, but which will force those trapped by it into the prevarications of "naturalness," and "canonicity."

The medicine that all must swallow is that if the Second Theorem crucially depends on technical choices for the form of *Con,* then the "philosophical implications" of the Second Theorem are masked behind a specially chosen set of mathematical conditions which are not generally justifiable, let alone justifiable within Gödel's form of mathematical skepticism. The Pyrrhonian "immanent" interpretation of Gödelian epistemology shows how arbitrary the invocation of an intensionally correct representation of provability is; it is not to be found in the dogmatic set-up, and this is entirely independent of the meaning of Hilbertian finitism, the truth of Russell's axiom of reducibility, or other special philosophical starting-points. There is nothing in the epistemological stage-setting for Gödel's proofs that allows such a move. The philosopher Gödel needs to argue like a true Pyrrhonian skeptic, and he can to a point; this is the basis of his mathematical criticism. But he is forced to shift gears, and compromise his position as his proofs develop—over the years! This surreptitious volte-face, effected in the transition from philosophy to mathematics, supports the idea of a "natural" choice of proof-predicate and consistency statement. But the price to be paid for this ahistoricism is a descent into epistemological chaos, because the epistemology underlying the mathematics is classic Pyrrhonism.

What is illustrated here is a historical conflict between mathematical progress and philosophical origins. The "deviancy" of Rosser sentences is a mathematical result which does not quite mesh with the elementary skeptical epistemology needed to work through the Incompleteness Theorems. But it is better to be historically honest and let the standard "philosophical consequences" of Gödel's work be somewhat weakened, than to obfuscate an important and still-evolving mathematical theory. It is heuristically unsound to replace an *interesting nexus of theoretical difficulties* with *apparently* ad hoc prescriptions that cover up good problems.[31] *Gödel changed a philosophical problem even as he improved it into a mathematical problem.* Ahistorical mathematical philosophy does not recognize this mathematically important fact. It is not that consistency might "really" be mathematically provable, but that the old notion of consistency was too naive to withstand the theoretical development that Gödel initiated. To recognize this historical event is to diminish somewhat the force of Gödel's work against the old foundationalists, but that at least acknowledges that Gödel began a new branch of mathematics by partially jettisoning an old set of philosophical standards. I agree, *Bew* "better expresses" provability than Rosser's *Prov*, but because it makes a mathematical theory work in a desirable way, and not because of a non-existent "accurate translation."

Indeed it is the complexity of the problem of translation between informal and formal mathematics that is at the heart of my crude and potted version of an intriguing and important tale in the history of twentieth-century mathematics. The historical process shows that the Second Theorem is not a theorem in the ordinary sense. It stands exactly on the border between informal mathematics—the mathematics that appears in journals, on chalkboards, and in notebooks—and formal mathematics, which is an imaginative abstract entity like a triangle, or manifold, or topological space. If one ignores the translation that takes place here between the informal and formal domains, there is no exclusion of consistency monsters. On the other hand, bringing in the Hilbert-Bernays conditions changes the original conditions for carrying out Gödel's proofs, from "working inside" a given formalism, to working inside the formalism *with some additional choices being made that are justifiable only after certain results are proved about the formalism and reflected upon with heuristic and historical hindsight.* The justification is perfectly sound, let there be no doubt; but the issue here is how that justification is introduced into a mathematical argument and how the argument then is changed. The process shows that, at least here, mathematics cannot be *completely* identified with its formal, metamathematical representation.

That is, I suggest, what Gödel's proof proves. It would not be too surprising if, as Plato says, "this most exquisite feature" of Gödel's work is transmitted down into the body of traditional mathematics. Like the "practical" applications of undecidability mentioned earlier in this essay,[32] we may see in the near future explicit applications of intensionality in the development of mathematical (and not just metamathematical) theories.[33] Such an event would mark the persistence of Gödel's genius and the triumph of his legacy: mathematical skepticism in an age of postmodernism.

Applied Decision Analysis, Menlo Park, California

NOTES

1 In their introductory essay on Gödel's *completeness* theorem in Gödel's *Collected Works*, Burton Dreben and Jean van Heijenoort point out "that to raise the question of semantic completeness the Frege-Russell-Whitehead view of logic as all-embracing had to be abandoned, and Frege's notion of a formal system had to become itself an object of mathematical inquiry and be subjected to the model-theoretic analyses of the algebraists of logic". Cf. S. Feferman et. al. eds., *Kurt Gödel: Collected Works Volume 1* (Oxford: Oxford University Press, 1986), p. 45.

2 That is, Gödel's argument can not be carried out without axioms for both addition and multiplication, essentially because Gödel-numbering requires exponentiation.

3 It was Gödel who originally suggested that the new axioms investigated by set theorists could be justified by their lower-order mathematical consequences in "What is Cantor's Continuum Problem" (1947), reprinted in P. Benacerraf and H. Putnam eds. *The Philosophy of Mathematics* Second ed. (Cambridge: Cambridge University Press, 1984).

4 Jeff Paris and Leo Harrington, "A mathematical incompleteness in Peano arithmetic," in J. Barwise ed. *Handbook of Mathematical Logic* (Amsterdam: North-Holland, 1977), pp. 1133-1142.

5 While it once was hoped that proofs of mathematically interesting theorems following from the new axioms (such as measurability of all sets of real numbers in the case of the axiom of determinateness, the continuum hypothesis for constructible sets, etc.) would help mathematicians decide in favor of one set theory or another, this has proved to be a false hope.

6 Sextus Empiricus, *Outlines of Pyrrhonism*, (trans. R. G. Bury), Loeb Classical Library (Cambridge, 1939), pp. 163-165.

7 This generality has been noted by Robert J. Fogelin in "The Tendency of Hume's Skepticism," in M. F. Burnyeat (ed.), *The Skeptical Tradition* (Berkeley: University of California Press, 1983), p. 398.

8 Isostheneia requires the construction of two alternative interpretations of the "appearances" dealt with by the skeptic. The undecidable sentence does not do this immediately, but Gödel himself points out in a review of work of Skolem's that his 1931 incompleteness work established that the natural numbers with addition and multiplication could not be represented up to isomorphism by a recursive set of axioms. So insofar as the natural numbers as a model is supposed to be the "reality" captured by (say) the Peano axioms, the incompleteness theorems show that this cannot be the case. (The proof is simply to note that the Peano axioms are consistent with the undecidable sentence and its negation, thus giving rise, by an application of the completeness theorem, to two non-isomorphic, because non-elementarily equivalent, models.) Cf. *Collected Works*, p. 379, for Gödel's review of Skolem, as well as the introductory remarks by Robert Vaught on p. 377. For arguments that skeptical appearance refers not to sense impressions but to a methodological aspect of the skeptical treatment of *any* kind of knowledge see M. F. Burnyeat, "Can the Skeptic Live His Skepticism" in M. F. Burnyeat ed., *The Skeptical Tradition* (Berkeley: University of California Press, 1983), and Michael Fried's review of Charlotte Stough's *Greek Skepticism*, in *Journal of Philosophy 70* (1973), pp. 805-10. This interpretation of skeptical appearance as being completely general is crucial to my entire discussion.

9 Jacques Derrida originally suggested in *Dissemination* trans. Barbara Johnson (Chicago: University of Chicago Press, 1981), p. 219, that deconstructive technique had an *analog* in mathematical undecidability, and because one often hears "undecidability" bandied about as a popular epithet, I thought it worthwhile to frame this essay within the context of the postmodernism debate. It might therefore legitimately be asked why I use the term "postmodern" as opposed to "poststructural" or "deconstructive." The answer is that postmodernism describes a general condition of knowledge, whereas deconstruction, as one moment of poststructuralism, self-describes a methodology or set of tropes whose practice contributes to a "postmodern condition." In this essay I take some broad features of postmodernism, identify their presence in mathematics, and trace that condition of mathematical knowledge to a set of skeptical arguments which also figure prominently in deconstructive technique. As I will suggest below, the relevant critique of postmodernism in our mathematical context is mathematical historicism.

10 Cf. Gisela Striker, "The Ten Tropes of Aenesidemus," in *The Skeptical Tradition*, p. 98.

11 *Against the Mathematicians,* (Loeb Classical Library Volume II), p. 239.
12 *Ibid.*, p. 87. This is Charlotte Stough's correction to Bury, noted in her *Greek Skepticism* (Berkeley: University of California Press, 1969), p. 58.
13 In "The skeptic's two kinds of assent and the question of the possibility of knowledge" in R. Rorty, J. B. Schneewind, Q. Skinner eds. *Philosophy in History,* (Cambridge: Cambridge University Press, 1984) pp. 257-258, Michael Fried writes on that aspect of skepticism that is best described as *immanent critique,* or the development of critical arguments using logic and concepts solely derived from the opponent's position. The meaning of Gödel's work for the Hilbert program depends very much on Gödel having been able to maintain such a stance within his proofs; at the same time this practice has almost descended into vulgarity by popularizers of deconstructive method.
14 *Against the Mathematicians,* p. 150.
15 *A Treatise of Human Nature,* Selby-Bigge edition (Oxford: Clarendon Press, 1888), pp. 267-68.
16 *Ibid,* p. 186.
17 This quotation from Pascal and the following from Arnauld and Hume are cited in Richard Popkin, "David Hume and the Pyrrhonian Controversy," in R. A. Watson and J. E. Force eds., *The High Road to Pyrrhonism* (San Diego: Austin Hill Press, 1980), pp. 133ff.
18 Popkin's discussion of "schizophrenia" in Hume is in "David Hume: His Pyrrhonism and His Critique of Pyrrhonism" in *The High Road to Pyrrhonism,* pp. 103-132. Gisela Striker's account of Arcesilaus' and Carneades' skeptical technique provides an analysis of ancient practices that come close to Humean schizophrenia. Cf. Striker's "Sceptical Strategies" in M. Schofield et. al. eds., *Doubt and Dogmatism* (Oxford: Oxford University Press, 1980) pp. 54-83, e.g. for why ". . . Cicero is perfectly right when he follows Clitomachus in thinking that Carneades advocated opinion only for the sake of argument . . . as far as Carneades himself was concerned, it seems most likely that he remained uncommitted both with respect to the framework in which the problems of his day arose, and with respect to the solutions he himself or others happened to offer." The similarity here to deconstructive method is striking. As for the mathematician, he is a skeptic in the ancient meaning of the term—an *inquirer,* who continues his perpetual investigations into knowledge, always searching and never finding.
19 *Dialogues Concerning Natural Religion,* Norman Kemp-Smith ed. (London: Thomas Nelson and Sons, 1947), p. 219n.
20 Cf. note 8, above, on Burnyeat.
21 On self-reference in Pyrrhonism see M. F. Burnyeat, "Protagoras and self-refutation in later Greek philosophy" *Philosophical Review 85* (1 Jan. 1976), p. 53: "the notion of peritrope or self-refutation is the notion that what the Skeptic says is falsified by his saying it, where his saying it is inclusive of, not—as it would be in a present-day discussion of self-refutation—exclusive of, the reasoning with which he supports his position." Of course this describes deconstructive practice with a vengeance, but the referential pointing-back-to earlier stages of argument by the skeptic is also central to Gödel's proof of the Second Theorem, as I discuss in the text. My remarks on the peritrope and oral debate are here based on Burnyeat's work.
22 This breaking up of an argument into stages is characteristic of skeptical *sunerotesen;* cf. G. B. Kerford, *The Sophistic Movement* (Cambridge: Cambridge University Press, 1981) p. 84.
23 In setting out to prove the Second Theorem, Cohen, in describing the full and final arithmeticization says that "this is perhaps the most subtle point in the entire subject and the reader is urged to stop and *review carefully what we have done up to here*" (my emphasis); Paul J. Cohen, *Set Theory and the Continuum Hypothesis* (Reading: W. A. Benjamin, 1966), p. 42. See also note 22 above on skeptical *sunerotesen.*
24 Cf. the references to Mostowski and Turing in Feferman in the paper cited in note 27, below.

25 Consider, for example, Theorem 3 on p. 240 of G. T. Kneebone *Mathematical Logic and the Foundations of Mathematics* (London: Van Nostrand, 1963): "If *K* is any simply consistent recursive class of formulae in [formal system] *P*, there does not exist any proof, formalizable in *P*, of the consistency of the system which results when the formulae in *K* are adjoined to *P* as additional initial formulae." This statement has the virtue of highlighting the fact that a translation is involved—"any proof" must refer to informal proofs—but in so doing it loses complete validity.
26 Kreisel has discussed the problem of translation *vis à vis* the incompleteness theorems in several publications, e.g. "Mathematical significance of consistency proofs" *Journal of Symbolic Logic 23,* (2 June 1958), esp. p. 177; "A survey of proof theory" *Journal of Symbolic Logic 33,* (3 Sept. 1968), esp p. 323. Kreisel's short note on Henkin's problem is "On a problem of Henkin's" *Indag. Math 15* 1953 pp. 405-6. Kreisel's views, which may derive from his interest in the later Wittgenstein, have been largely ignored, but they may return if intensional contexts become a practical issue, e.g., in the semantics of programming languages. The philosopher of mathematics who has written most on the problem of translation is Imre Lakatos in J. Worrall and E. Zahar eds., *Proofs and Refutations: The Logic of Mathematical Discovery* (Cambridge: Cambridge University Press, 1976), esp. Chapter 2.
27 Cf. S. Feferman, "Arithmetization of metamathematics in a general setting" *Fundamenta mathematicae 49,* pp. 35-92. Feferman also provides a list of results he classifies as intensional or extensional. The result is "*extensional* if essentially only numerically correct definitions are needed, or *intensional* if the definitions must more fully express the notions involved, so that various of the general properties of these notions can be formally derived" (p. 35).
28 *Ibid.*, pp. 67-8 (my emphasis).
29 *Ibid.*, p. 79.
30 George Boolos, *The Unprovability of Consistency* (Cambridge: Cambridge University Press, 1979) p. 137 (my emphasis).
31 Boolos himself remarks that difficult theoretical problems with Rosser sentences may turn on the "(awful? attractive?) possiblity that . . . the answers . . . may depend on the way sentences of arithmetic are assigned Gödel numbers." *Unprovability,* p. 139.
32 See notes 3, 4 above.
33 Joseph Stoy describes an advantage of denotational semantics as its clear demarcation between the intensional realm of programming languages and their extensional interpretation in the partially ordered sets of the Scott-Strachey theory. Cf. his *Denotational Semantics: The Scott-Strachey Approach to Programming Language Theory* (Cambridge: MIT Press, 1977).

A specific result leading in the same direction is the theorem of Baker, Gill, and Solovay showing that the relativized P = NP conjecture in computational complexity can be made true *or* false for suitable choices of the base language. This remarkable result shows, for some, that a solution to the unrelativized P = NP may demand a new kind of mathematics since all known methods to settle the conjecture would relativize between languages, in contrast to this theorem suggesting the intensionality of P = NP. Cf. T. Baker, J. Gill, and R. Solovay, "Relativizations of the P = ?NP question" *SIAM J. Comput. 4,* pp. 431-442.