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Dialectic and Diagonalization

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This essay is about mathematics as a written or literate language. Through historical and anthropological observations drawn from the history of Greek mathematics and the oral tradition preceding the rise of literacy in Greece, as well as considerations on the nature of alphabetic writing, it is argued that three essential linguistic features of mathematical discourse are jointly possible only through written, alphabetic language. The essay concludes with a discussion of how both alphabetic principles and issues related to literacy faced by the Greeks in the axiomatization of geometry play a central role in some specific metamathematical theories. Drawing extensively on the work of Árpád Szabó, Eric Havelock, and Albert Lord, the implications developed between Szabó's history of Greek mathematics and Havelock and Lord's theories of writing and oral traditions (Homer's in particular) are the author's own, as are the applications to modern logic.

> What say you of our human constitution? How is this human race of ours endowed?. . . is it not very like the diagonal?
>
> The Eleatic Stranger, Plato's *Statesman*

I

In Don DeLillo's remarkable comic novel of mathematical ideas, *Ratner's Star*, there is a section entitled 'Rob Talks in Quotes', about Robert Softly, logician and mentor of the book's hero, the young mathematician Billy Twillig. DeLillo writes that Billy

> didn't like his mentor's very occasional tactic of pronouncing certain words as though they warranted quotation marks. The practice seemed to have a source deeper than mere sarcasm. Softly sometimes employed this vocal rebuke, if that's what it was, in circumstances that appeared to be completely unsuitable. He would refer to a table, for example, as a 'table'. What sort of inner significance was intended in such a case? It was one thing for Softly to use a sprinkle of emphasis when speaking of someone's 'need' for 'rest'. But when he put quotes around words for commonplace objects, the effect was unsettling. He wasn't simply isolating an object from its name; he seemed to be trying to empty an entire system of meaning.[1]

Ratner's Star tells the story of a secret project to define the imaginary metamathematical language Logicon, through which a message from the cosmos, thought to originate at Ratner's Star, can be decoded and interpreted. The Logicon project is never completed and *Ratner's Star* turns out to be an esoteric history of mathematics as told by a bard, replete with secret allusions to mathematical history and mathematical concepts. Most of these allusions are to modern mathematical logic, including the above disquieting reference to a fundamental method of contemporary logic known as Gödel-numbering, named after its inventor, the mathematician and logician Kurt Gödel. Gödel-numbering is the formal technique of mathematical quotation, and is the primary mathematical tool through which mathematics is used reflexively to develop mathematical theories about the relative power and limitations of mathematical definitions, axioms, proofs, and theorems themselves.

DeLillo's insightful joke is that it is impossible in practice for Robert Softly to 'talk in quotes', and it is a mark of DeLillo's mathematical sophistication that he frames the effect as unnerving. The humorless mathematician might react that Gödel-numbering, as mathematical quotation, is a mathematical function like any other, a 'mapping' from numbers to numbers, like addition or multiplication, whose particular interpretation is that numbers are thought of as coding linguistic types: variables, logical connectives, formulas, sentences, axioms, postulates, and so on. Such a quotation function may apply to speech as well as to the inscriptions on a bone, or the electronic signals carried by a computer. To misunderstand this point is to confuse (meta)mathematical abstractions with their intended interpretations. But is it really so simple to demand or postulate that quotation is 'just more mathematics'?

DeLillo's parodic notion is better revealed if we refer not to *speech*, since this connotes a kind of verbalized text, but rather to *orality*, or of language embedded in a form of life entirely without the written word and its associated technology of preserved communication. In such a form of life, language does not fossilize, and the transmission of knowledge between individuals, communities, and generations is not subject to anything like the relative norms of constancy and intersubjective objectivity possible through writing – whether by alphabet, ideograms, syllabary, or a mixture of all three. Even for the mathematical platonist, nominalist, or intuitionist, it is reasonable to ask: How do we imagine that metamathematical notions such as Gödel-numbering and self-referentiality are to be realized in an oral mathematics – Are such theoretical constructs possible within a form of life dominated by the ear rather than the eye, and ephemeral voices rather than permanent texts? Are we certain that we would have no difficulty, in principle, in 're-proving' in an oral mathematics classical results of modern logic, such as the famous Gödel Incompleteness Theorems on the limitations of mathematical systems? Could one argue about 'sentences' referring to theorems and, as in Gödel's ingenious constructions, 'saying of themselves' that they are unprovable? If not, then it may be because logic is not just mathematics, but is naturally interpreted mathematics whose validity depends on the literate form of life in which logical theorizing makes its home. 'Mathematics is just a language' is a shopworn piece of good advice, but what *kind* of language we may ask, literate or oral? This is the exquisite problem that Don DeLillo has so wittily posed for his readers.

II

The value in questioning whether mathematical proof is peculiar to literate cultures is heuristic. The question prompts us to look for some basic properties of language and mathematical discourse necessary for proofs to be systematically carried out at all. After all, all it takes to be a mathematician is to prove theorems, meaning to make assertions going quite beyond Babylonian or Egyptian calculations without proofs, however ingenious or useful. And not in your head, but externalized in language, even if mechanically produced by computer. Conversely, every mathematician has had to externalize his ideas. Fermat will never receive credit for the proof of the still unproved conjecture that goes by his name and which was 'too long for the margin' of his book. A necessary and sufficient condition for mathematical theory is the production of mathematical discourse, whether of notebooks, journals, or chalkboards. What are the basic properties of the practice of conjecturing and proving theorems?

Three properties of mathematical discourse appear to be fundamental. Mathematical discourse should be *context-free*. Because mathematics produces theories of timeless, non-empirical entities such as numbers, points, lines, triangles, and manifolds, the interpretation of mathematical expressions should minimally depend, if at all, on guesses or cues needed to decide ambiguities of meaning or expression. A timeless discourse of timeless concepts requires a timeless language. This does not mean that there are or need to be absolutely timeless concepts to legitimate mathematics, but only that we have a generous and flexible discourse with the properties and techniques needed to develop theories of timeless entities. Indeed modern mathematics is the actual realization of the narration of form repeatedly described by Plato in the *Republic* as the goal of philosophy – regardless of whether there really are metaphysical forms corresponding to mathematical objects.[2] To be a mathematician is to speak and write as a virtual platonist, whether or not any *things* are mechanically intended as her theories' referents.

There is a special case of context-free discourse, worth singling out on its own. Mathematical discourse should be *autonomous*, as expressed by Andrew Hodges in his biography of the logician Alan Turing. Turing, writes Hodges,

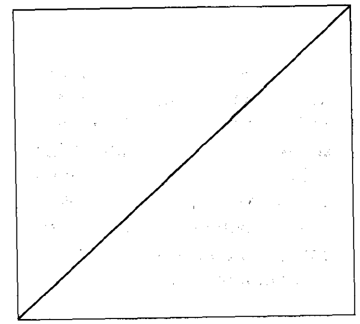
has immortality through the expression *Turing machine*. Many people must have used his name without any conception of his historical existence – the nearest thing to the life as a disembodied spirit that [Turing] once pondered on. Going even further, modern papers often employ the usage of *turing machine*. Sinking without a capital letter into the collective mathematical unconscious (as with the *abelian group*, or the *riemannian manifold*) is probably the best that science can offer in the way of canonisation.[3]

Mathematics, as Karl Popper says, is a form of knowledge without a knowing subject: it is completely irrelevant to mathematical practice to know the social or biographical genealogy of a mathematical proof, or to learn a proof from a certain individual. The beauty of appellations such as 'Fermat's last theorem', or 'Gödel-numbering', is that original intent is completely erased through expressions referring to anything but Fermat or Gödel. Being a student of David Hilbert or John von Neumann bears no analogy in this respect to having been psychoanalyzed by Freud.

Finally, mathematical proofs and theorems are *referential*, meaning that theorems and proofs are objective, intersubjectively available linguistic constructs with well-defined criteria of identity and difference: *mathematical theories are things*, full of virtual objects made possible through language. For most practical, if not historical purposes, mathematicians easily differentiate theorems and decide whether they prove the same result, or use equivalent concepts. There are original proofs and variants, lemmas and corollaries, conjectures and counterexamples, all because mathematical discourse is expressed through language allowing mathematicians to refer to these classes of assertions.

The point of these three rough distinctions is not to define mathematical discourse, but to note some obvious properties without which it would be difficult to develop and express proofs. Without referentiality, there would be no well-defined beginnings or ends to theorems and their proofs, no differentiating lemmas from corollaries, and generally no means of making the essential demarcations that delimit levels of mathematical knowledge and help define criteria of mathematical truth. Without autonomy or more general forms of context-freedom, criteria of mathematical knowledge would necessarily incorporate some kind of strange *Verstehen*-proof, in which readers of proofs would, like the interpreters of a historical tradition, play an explicit role in the process that creates the evidence supporting a claim to truth, and judge the claim on this constitutive basis, rather than as a detached judge of objective standards.[4]

III



The most important of ancient mathematical proofs based on deductive, logical argument is the proof of the incommensurability of the side of a square with its diagonal: that the ratio of the magnitudes corresponding to the square's side and its diagonal can *not* be expressed as a ratio of whole numbers $a{:}b$. In a modern sense not current among the Greeks, we say that the length of the diagonal of a unit square, $\sqrt{2}$, is *irrational*.[5]

The proof of this theorem dates from the mid-5th century BC, somewhat less than three hundred years after the completion by the Greeks of the world's first true alphabet, and toward the end of the centuries-long diffusion of literate practices between Homer and Plato. The theorem is proved indirectly, or 'by contradiction', meaning that one assumes that the diagonal *can* be expressed as a ratio $a{:}b$, and then shows that a contradiction ensues – namely that 'some number a is both even and not even'. The earliest known examples of deductive proof are from Pythagorean arithmetic, including theorems such as 'even plus odd is odd', and 'odd plus odd is even', and so forth. The key to the proof of the incommensurability of the square's diagonal with its side is to reduce that problem to one about even and odd numbers, and *there* to make use of logical contradiction. Our understanding of the origins of deductive proofs is hindered by the absence of surviving mathematical texts dating from before Euclid, who compiled his *Elements* around 300 BC, a hundred years after the death of Socrates. However, there were many deductive proofs invented by the pre-Socratics, especially the poet Parmenides and his student Zeno, well-known for his several 'paradoxes' – though their non-mathematical proofs were not about numbers or geometrical objects, but about other abstractions, such as Being, the One, motion, and time, much as modern mathematics is often about non-numerical or non-geometrical entities.

Here is a reconstructed example of such a proof. Assume that what is came into being. Then what is came from what is, or came from what is not. If what is came from that which is, then that which is was already in existence, and did not come into being, which contradicts our assumption. If what is came into being from that which is not, then what it once was what is not, and could not have come into being; again a contradiction. Hence to demand 'what is came into being' leads to a contradiction; the

Greeks would cautiously infer only that its negation is 'less contradictory than its opposite', though today we might claim to have proven that what is did not come into being.[6] Like an orthodox proof, a Parmenidean proof about Being or the One has no subject of action. The proof's principal verb is the timeless *to be*, with no distinction drawn between past, present, or future, a description intended of *x as such*, whether *x* is Being or the length of the diagonal. If one treats pre-Socratic philosophy with the same suspension of disbelief about the metaphysics of Being that we normally provide for the metaphysics of numbers – neither schoolchildren, accountants, von Neumann nor Gödel *have* to believe in the reality of numbers to do their mathematics – then Parmenidean philosophy is in the form of theorems and proofs, no less so than Euclid's.[7]

It is often recognized that the invention of philosophy, with its affiliated conceptions of intellectuals, experts, and elites by the pre-Socratics and Plato, was developed through a critique of Homer and the implicit epistemology of his oral world. But the mathematicians Plato revered and used as his model also participated in his ancient 'quarrel between philosophy and poetry', and mathematics also emerged from, and in implicit opposition to, Greek cultural norms and patterns of thought saturated by ancient poetic practice. Mathematicians were not leaders in this battle, but, by adopting Parmenides' method of indirect proof, they helped lay the technical foundations for a cultural revolution based on a critique of Homeric language and the forms of life supporting it. 'Mathematics is just a language' – In the oral discourse of preliterate Greece antedating Homer, a sustained tradition of mathematical theorizing is impossible in practice, and proof-techniques that would become paradigmatic of both Greek and modern mathematics are impossible in principle.

While as a matter of cultural history, a sustained and learnable discourse that is context-free, autonomous, and referential did not exist for Homer, Plato was able to point to mathematical reasoning as a model for philosophical speech embodying the dialectical methods illustrated in nearly each of his dialogues. To take the example of the *Theaetetus*, deduce a contradiction from the starting-point 'knowledge is perception', and you have proved that 'knowledge is not perception'. Mathematicians, already influenced before Plato by Eleatic philosophers such as Parmenides and Zeno, had provided their own substitute for the Homeric discourse of aggregated actions, paratactic descriptions and continual, contradictory change attacked in the *Republic*. But Socrates and Plato could use this method only as part of a *strictly negative* dialectic. For example, having proved that knowledge is *not* perception, nothing much else follows – whereas the mathematician, having shown that a number *is not* even, can say positively that it *is* odd – hence the inconclusive and suspended endings to so many Socratic colloquies. And because the Eleatics created and dealt

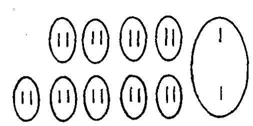
with abstractions like Being, they could implicitly invoke a law of excluded middle – that either Being *is or is not*, or what we call *tertium non datur* – without appearing arbitrary; Plato, arguing over common terms like justice, virtue, and knowledge, had no such option.

There is then a historico-cultural paradox at work here, epitomized by Socrates' admiring remark to Parmenides that, while 'Zeno has many weighty proofs to bring forward . . . you assert, *in your poem*, that the all is one, and for this *you advance admirable proofs*'.[8] The paradox is that Parmenides was both the original superlogician and a poet, a transitional figure bridging the poetic world Plato despised and the mathematical world he simultaneously admired and imitated so well. Evidently, there must then be some essential relationship between orality and mathematical proof, and consequently between mathematical proof and literacy, or what orality is not.

IV

What is essential to the proof of the incommensurability of the side of a square with its diagonal is that it is impossible to demonstrate the existence of incommensurable magnitudes *empirically*, with pebbles, sand drawings, or otherwise. This proof cannot be convincingly suggested by a picture, it must be arrived at by an act of intellect. Any real, measurable length can be approximated arbitrarily well by ratios of whole numbers, so an incommensurable 'length' of the diagonal makes no sense on empirical criteria. This new abstraction can only be demonstrated with rules of proof. The importation of Parmenidean dialectic into empirical mathematics based on heuristics and arithmetic calculi occurred by replacing visual-empirical rules of evidence, such as collecting groups of pebbles or drawing geometric figures, with rules for bringing stories to a conclusive ending based on the new method of logical contradiction. Hence the revolutionary nature of the proof of incommensurability.

Among the alternatives to deductive proofs, including proofs by contradiction, are quasi-empirical proofs like those discussed in Plato's *Meno*, in which one *shows* a mathematical principle by proving it visually for a generic instance. For example, the arithmetic principle 'Odd + Odd = Even' can be shown true by a picture such as:



Deiknymi is the Greek word for proof and (in the appropriate verb form) was used like our 'QED' to signal the end of a proof, but in early methods of proof the 'showing' or 'pointing out' which *deiknymi* meant was literally that: the 'proof' was obtained by reflecting upon what was shown, pointed out, or seen.[9] Only after mathematicians developed anti-visual and non-empirical techniques did *deiknymi* come to have its modern, intellectualist meaning and could the transformation of *that which is shown* into *evidence* be eliminated from mathematics altogether. Hence, while to our ears definitions such as Euclid's 'line is a breadthless length' may have an anachronistic ring, it substituted well for Plato's 'that of which the middle covers the ends', as if describing how a line looks to an eye placed at one end and looking along it.[10] In indirect, Parmenidean proofs, such as that for 'what is cannot have come into being', or Zeno's proofs of the impossibility of motion, time, or space, no appeal is made to visual or empirical evidence. While the Eleatics *had* to be anti-empiricist, because the conclusions of their own proofs left no alternative, mathematicians were free to adopt the method of indirect proof as means of legitimating concepts, such as the diagonal, which had no empirical correlate. Thus Heraclitus' attack on Homeric polymathy, Parmenides' arguments against the 'many', and other pre-Socratic polemics were reinforced by a technical need in mathematics for abstract proofs. Eliminating empirical appeals from mathematics therefore represented, as a kind of corollary, a major triumph for anti-Homeric criticism. But as we shall see below, the problems of empiricism and the stain of the specific returned to haunt mathematicians in the mathematization of geometry, and may recur in contemporary logic.

So the revision of Greek mathematicians' conceptual rule-book occurred when it was discovered that the diagonal represented a new type of conceptual entity which could not be dealt with adequately by existing criteria of mathematical truth. In spite of the fact that key mathematical methods grew directly from sources with significant ties to oral traditions, such as Parmenides' poem, this major change in mathematical method was readily assimilated into a literate culture, along with the other fragments joining together to form the Greek intellectual mosaic. *De facto*, Greek mathematics developed as part of an attack on a sophisticated oral tradition and is at home only in the alphabetic literacy that came in Homer's wake. This inter-embedding of perceptual and communicative modalities is a general feature of Greek conceptual history, but is fundamental to the relationship between mathematics and alphabetic literacy, because the central innovation of the alphabet – which, like mathematical proof, was created by the Greeks – is the successful use of consonants as a complete abstraction of speech: a true non-empirical notion, like the diagonal of a square.

V

The Greek alphabet, for the first and only time in recorded history, introduced an elementary and essentially complete system of structured names for the sounds of speech. Earlier writing systems either tried to map symbols one-to-one to sounds (these are syllabaries), or, as in Phoenician, indexed only limited combinations of sounds, such as consonants plus vowels: *be, ba, bo, bu, da, de, do*, etc. Pre-Greek systems *copy* sound, while the Greek alphabet *analyzes* linguistic sound into two components: the vowel as a column of air and the consonant as a start or stop by the lips, palate, tongue, and teeth. The list *be, ba, bo, bu* may be captured by rules producing forms that may be written consonant + vowel$^{\text{short/long}}$. To use a mathematical analogy, sounds are represented as xy^j, where x represents a starting consonant, y is a vowel, and j stands for the shortening or lengthening of the vowel. The Greek accomplishment was to complete the analysis by developing a set of vowels allowing all needed combinatory forms, such as $w^i x^j y^k$, where w is a consonantal start, x is a vowel, y is a stop, and so on.[11]

The innovation of the Greek alphabet is to realize the complete potential of the abstract notion of a consonant as a non-sound. Like the length of the diagonal, the consonant is a non-empirical concept. It is not possible to have a 'start' or 'stop' in physical isolation, each must be accompanied by a rushing column of air. The consonant 'S', says Plato in the *Theaetetus*, is 'nothing but a noise, like the hissing of the tongue, while B not only has no articulate sound but *is not even a noise*, and the same is true of most of the letters'.[12] Consonants are meaningful only in conjunction with the vowels which they modulate. The alphabet is a true analysis of the sounds of speech into constituent elements (as the Greeks called letters), most of which have no empirical counterparts like the sounds $\{s_1, s_2, \ldots, s_n\}$ represented by a syllabary. True alphabetic writing like the Greeks' is a Cartesian analytic table; syllabaries are incomplete, partially structured lists of speech sounds. As lists, syllabaries are homologous to the aggregated, paratactic actions characteristic of Homer, while the alphabet makes possible the conditions of context-freedom, autonomy, and referentiality needed for mathematical proofs. The consonant as a separable unit is the linguistic analogue not only of the mathematical diagonal, but the many other conceptual abstractions which emerged between the pre-Socratics and Aristotle: space, dimension, body, matter, void, change, motion, psyche, etc. Whereas for the Homerics, paratactic heuristics guided explanations in the form of actions succeeding actions in running sequence, these new timeless abstractions, in coordination with the novel use of a timeless *to be*, helped to create and enhance the possibility of subordinating effects to causes, and the substitution of general descriptions for particular instances and the performative statements expressing them.

By providing a simple, economic, and well-defined mapping from combinations of symbols to sounds of speech, the alphabet eliminated the need for many contextual determinants provided by speech-acts. The alphabet's power lay in its economy of means for unambiguously representing large fragments of speech with a minimum of symbolic machinery. With a finite syllabary $\{s_1, s_2, \ldots s_n\}$ one is eventually limited by the contextual information needed to unequivocally decide on the sounds, and hence the meaning represented; the disambiguating power of the alphabet is absent except for the relatively small number of cases covered by a syllabary. In either an oral culture like Homer's, or a protoliterate culture employing a syllabary, there are few, if any, examples of what Parmenides and his followers mean by 'that which is or is not'. But the alphabet itself makes empirical speech into a concrete case of abstract thought. For example, Parmenides' atomist followers saw that their combinatory theories of matter were analogous to what the alphabet had done for speech, and that atoms were like letters.[13] Syllabic signs can both 'be' and 'not be', and the entire thrust of Plato's critique of the Homer world as perpetually 'becoming' makes little sense when one's very language might serve as a counter-alternative. A language implicitly obstructing such an argument became possible only through the alphabetization of Greek and the Greek mind, and Parmenides' invention of indirect proof might have remained an esoteric poetic device without alphabetic writing as a means to develop and to deploy it in other discursive forms. In any case, no *sustained* Parmenidean or mathematical logic is possible in a language requiring either the contextual data needed to make use of a syllabary, or the performative, case-specific cues needed in oral discourse generally.[14] In fact, to return to our earlier alphabetic notation $w^i x^j y^k$, it is noteworthy that this is precisely how logicians represent mathematical proofs as numbers: w,x,y,i,j,k are numbers representing the structure of a proof (actually one needs about a dozen numbers, but the principle is the same), one multiplies all the numbers together into a single code, the 'Gödel-number', and the final value uniquely and unambiguously represents a proof, just as the Greeks first discovered how to best represent sounds. This rediscovery of alphabetic literacy in modern mathematical logic is one of the keys to understanding the dependency of mathematics on literacy, and I shall return to it below.

The alphabet makes full-blown mathematical theory possible because one can be a real mathematician or a virtual Platonist only by having available a language supporting context-free, autonomous, and referential discourse. In an oral culture supporting speech-acts and a conceptual economy in direct antithesis to those needed for mathematics, fundamental linguistic practices which dominate the form of life create insoluble problems for expressing mathematical ideas. Insoluble, that is, without the introduction of an entirely new alternative for linguistic representation.

Consider, if you will, the notion of two theorems and their proofs being *the same*, say as one might judge two students' solutions to a problem, or the simultaneous discovery of two proofs of a mathematical conjecture. The proofs would naturally not be identical word-for-word, but mathematicians generally can reach a consensus about whether the two proofs differ in any fundamental respect, to what extent they really utilize different methods, whether one has more constructive content than the other, and so on. (Of course from the sterile viewpoint of outmoded formalist philosophies of mathematics, this issue has no content, but in fact to re-prove old results in new theoretical frameworks is a salient feature of modern mathematics.) Could similar differentiating practices be accomplished in a purely oral culture such as Homer's, in which general knowledge is carried from generation to generation through (as Eric Havelock calls the *Iliad*) the 'tribal encyclopedias' of oral poets and singers?[15] Perhaps we can imagine a theorem being stated as the opening or closing verses of a song, with the proof somehow embedded in the poetry coming before or after, like the mathematical rules contained in some Indian *sutras*.[16] Certain lines or characteristic expressions might mark stages of proofs, as do lemmas and corollaries, and two such proofs would be 'the same' for the same reasons that we describe the student's proof as being the same as the teacher's, yet 'different from' the proof in the text. But given what is known about oral traditions and the world of the Homeric poets – who left the stage of Greek education with the spread of literacy, and the inventions of philosophy and mathematical proof – one has to conclude that this scenario, like Don DeLillo's Robert Softly, is an unrealizable fiction. While the practical genius of archaic Greece is demonstrated by great achievements such as the *polis*, neither Homeric poets nor preliterate Greeks had the discursive techniques needed to distinguish an elegant proof from a complicated proof of the same theorem, or an algebraic proof from a geometrical proof, or a mathematical proof from a metamathematical proof. Nor could they prove theorems at all.

Homeric poets could not prove theorems because they had no means of segmenting language into conceptual units based on the categorical schema needed for mathematical discourse. 'Language' is not even available as an autonomous or virtual object independent of its expression in speech. This complete lack is put bluntly by Albert Lord in *The Singer of Tales*, when he says of the Yugoslav singers who he and Milman Parry (who first proved that the formulaic structure of Homeric verse was principally a mnemonic device for oral poets) lived with, that 'singers do not know what words and lines are' – *a fortiori* they could never have theorems and proofs:

> When asked what a word is, he will reply that he does not know, or he will give a sound group which may vary in length from what we call a word to an entire line

of poetry, or even an entire song. The word for 'word' means an 'utterance'. When the singer is pressured then to say what a line is, he, whose chief claim to fame is that he traffics in lines of poetry, will be entirely baffled by the question, or he will say that since he has been dictating and has seen his utterances being written down, he has discovered what a line is, although he did not know it as such before, because he had never gone to school.[17]

The sound groups of poets do not necessarily coincide with words, and hence 'any particular song is different in the mouth of each of its singers'. The first performance of a song is not an imperfect rehearsal or 'version', but is continuous with the repetitious process of teaching and performance that is inherent in maintaining oral traditions. There is no *variant* song because there is no *original* song to be varied, and indeed, 'our concept of "the original", of "the song", simply makes no sense in oral tradition . . . Author and original have either no meaning at all in oral tradition, or a meaning quite different from the one usually assigned to them'. Oral composition, creation, performance, and transmission are literate metaphors used to define, describe, and separate aspects of a unitary process, but the mathematician's need to distinguish precisely the steps of a proof, its beginning and end, have no obvious correlate in the oral world. All the referential notions required by mathematicians – linguistic identity, sameness, originality, variance, and difference – are not part of oral speech. Conversely, these are exactly the empirical, linguistic analogues of general philosophical notions about identity – especially that of originary form – introduced by Plato and the pre-Socratics in their critique of Homer. These linguistic relations then became intrinsic to alphabetized language, regardless of the problems and issues of Western philosophy, and made virtual Platonism into Western reality. Syllabaries also cannot support the context-free discourse needed by mathematics, as one always needs 'help' from scribal conventions or the contextual limitations imposed by carefully circumscribed(!) speech-acts, such as accounting practices, or ritualized recitation. Syllabaries can support limited metalinguistic categories, but without the power of the alphabet's analytic table of consonants and vowels one cannot get much higher in the referential order of language than words and sentences. Explicit criteria of identity and difference for such sophisticated notions as theorems and proofs are beyond the power of a syllabary that is practically useful.[18]

Not only referentiality but mathematical autonomy is similarly absent in many oral traditions. In Plato, the word *mimesis* captures the psychological immersion of both singer and audience in poetic performance.[19] Through complete identification with his song and its content, the poet maximized the use of the mental and physical resources needed to recall extensive verses from memory, unaided by a text. While dance or the music of a lyre could help the singer structure his poetic recall, his lyrical success depended very much on his ability to be psychologically overcome by his performance. Plato complains that poets 'effect their narration through imitation', meaning that he delivers the poem 'as if he were himself Chryses and tries as far as may be to make us feel that not Homer is the speaker, but the priest, an old man'.[20] The audience, which was not being 'entertained' in any modern sense, similarly abandoned themselves to song as the most efficient means of learning. 'Even the best of us [Acheans]', says Plato, in listening to Homer or some other great poet, 'feel pleasure, and abandon ourselves . . . and we praise as an excellent poet the one who most strongly affects us in this way.'[21] Poetic *mimesis* stood for this amalgam of composition, poetic immersion, performance, audience learning and audience identification happening all at once, and Plato rejects the Greek poets' lack of autonomy in the hypnotic identification of poet, audience, and song. But in an oral tradition no choice is possible if one wants to utilize speech best as a means of transmitting knowledge; personification by the poet and audience identification in song is simply the most efficient technology of communicating mores, political practices, ceremonial protocols, and other elements of tradition, through multifarious forms of performed speech: cult hymns, processional songs, marriage songs, funeral dirges, begging songs, elegies, etc.[22]

Plato more than anyone else elevated the idea of an integrated, autonomous self that could stand apart from tradition and examine it as an independent, autonomous entity. The paratactic 'me' of the Homerics, like the concatenated series of events and actions described in poetic song, shifts among an endless series of moods, either in poetic performance or audience listening, continually 'becoming', as Plato says, and in contradiction with itself. The required correlate of the autonomous language forming the basis for mathematical discourse is an autonomous, or virtual, subject who can distinguish itself from its source of knowledge. But for the Homerics, no such unified self existed, and there was no corresponding body of autonomous knowledge. As pointed out many years ago by Bruno Snell, the relevant concept of the body, for example, is as a paratactic assemblage of parts. There are 'limbs' filled with strength and connected at the joints, and 'skin' as the outer border or surface or figure of a man, but the skin does not envelop another anatomical substance. Even visual perception is described paratactically before Plato. There are words for 'to have a particular look in one's eyes', 'to have a noticeable gleam', or other specific visual attitudes, but no vision as such, the operation common to all 'seeing'.[23] *Theoria*, before it took on the meaning 'to look on', or 'to contemplate', did not denote the activity we now associate with intellection, but was derived from its noun form, meaning a 'spectator', such as one attending the games of a rival town. These examples can be multiplied, but their singular consequence is that there could be no autonomous mathematical theories for Homerics because *there were no experiences* through which 'intellects' could define autonomous entities like theorems and proofs. There is nothing in the Homeric individual's entire being that can be analogized to the dialectical mathematical practices that Plato would elevate to a philosophical norm.

VI

From the pre-Socratics to Plato, the intellectualist attack on Homer succeeded through the development and institutionalization of new linguistic norms and practices that were antiparatactic, favored locutions of timeless existence, and shifted the cultural 'ratio of the senses' (as Walter Ong calls it) from the ear to the eye. But the transition to abstraction and literacy was gradual, as shown by the dual status of pre-Socratic philosopher-poets such as Xenophanes, Heraclitus, and Parmenides. The development of mathematical proof similarly shows that mathematicians before Plato stood in two Greek worlds, one oral, transient, and protean, the other literate, intellectualist, and stable. In spite of the radical transformation in thinking aided by the alphabet and the democratization of literacy in Greece, some of our most basic notions about the structure of mathematics, axiomatics, and deductive proofs are literate transmogrifications of language-games grounded in a predominantly, but not exclusively, oral tradition.

Parmenides' invention of indirect proof, or proof by contradiction, is his greatest contribution to science, and is by far the most important method of mathematical proof. Take away the method first expressed (to our knowledge) by Parmenides in the song addressed to his student Zeno, *The Way of Truth and Opinion*, and the body of mathematical knowledge is decimated. But the method requires additional structure imposed on the beginnings and ends of proofs. Plato's *Parmenides*, for example, is parodic in its circular repetitions of pure Parmenidean proof, with Parmenides' endless returns to new beginnings. Indirect proof was successfully introduced into mathematics only by translating other stylized oral and rhetorical practices into literate abstractions, including the fundamental concepts of *axiom* and *postulate*, which were originally dialectical terms applying to a debate between two speakers.[24] *Hypothesis* or *postulate* signified a request or demand to provisionally accept an initial proposition for discussion, like 'knowledge is perception', and so many other of Plato's tentative propositions which the interlocutor proposes as an opening conversational gambit. Typically there was some level of common agreement, but using a postulate did not commit one to the truth of the original statement. Socrates asks often that he 'be allowed' to make use of hypotheses; these are then conceded or agreed upon by all parties; then consequences 'follow', literally, as following in conversation; and finally, perhaps, the conversation ensues in a 'contradiction', signaled by the disagreement between two conversants and a return to the falsified postulate. *Axiom*, unlike hypothesis, was used for an explicitly *false* demand, and was accepted as a starting-point only with reservation. For example, to begin a debate with an Eleatic philosopher assuming that 'Being can be many' would be to start with an axiom. In a remarkable turn taken by the history of ideas, postulates and axioms have today, at least in colloquial usage, taken on exactly the opposite meaning in 'foundational' mathematics from what they had when mathematics was a quasi-dialectical discipline, employing speculative abstractions such as the One.

In Euclid, postulates include geometrical assumptions related to motion, such as *connecting* a line between points, or *drawing* a line from a point to infinity. Parmenides and Zeno had proven that the concepts of motion and becoming led to contradictions, but in spite of their conceptual refutation, they none the less regarded empirical motion as real (just as, conversely, we do not believe in physical, actual infinities, though they abound in mathematics). Otherwise put, a science of space was *necessarily* based in sense perception. Geometrical postulates, as quasi-empirical assumptions involving hypothetical 'drawing' or 'connecting', were consistent with the use of Parmenidean logic, even if in contradiction with Eleatic philosophy at some more general level. However, the Eleatics did not condone unqualified assertions about equality, 'is', or 'being'. The indivisibility, homogeneity, and superconstancy of Being and the One formed the basis for Eleatic philosophy, and every argument would be mobilized to prevent the 'division' of the One into a Homeric 'many'. This *raison d'être* of Eleatic logic led, for example, in mathematics, to disallowing fractions because their existence implied that the One, as a mathematical unit, was divisible and hence 'many'. The unit is *not even considered a number*, since numbers are 'multitudes' of units – and so there are awkward formulations of conditions like 'If x is a number *or one* . . .'.[25] Fractions were used in Greek business and commerce, as they had been in Egypt and Babylonia, but for the Eleatics they were theoretically unsound (but could be treated as ratios $a{:}b$). For mathematicians, 'the many' had evolved from a codeword for the Homeric world-view to a set of theoretical heuristics influencing mathematical theory. So for mathematicians to utilize *general* rules for equality, such as symmetry ($a = b \rightarrow b = a$) or transitivity ($a = b$, $b = c \rightarrow a = c$), such assumptions had to be invoked as axioms because, in Eleatic philosophy, to say '$a = b$' implies '$b = a$' was contradictory. A thing was only the same as itself, not some 'other thing', and '$a = b$', so interpreted, was as self-contradictory in the ordinary, non-symbolic prose shared by mathematics and philosophy as 'One is divisible', or 'Being is not'. Introducing equality rules as other than axioms would have violated the spirit and letter of Eleatic method. In any event, axioms and postulates were dialectical methods needed to structure mathematics as a deductive science, and aided in the transformation of mathematics from an empirical and algorithmic subject to a theoretical discipline. Abstract mathematics before the time of Plato was in this respect guided by oral methods and the performative dialectics of a craft-literate culture.

According to Plato, it was difficult for many to be clear about the new, anti-empirical, intellectualist foundation for mathematical geometry. Locutions such as 'squaring' or 'cubing', says Plato, were misleading because geometry was not supposed to have any relation to real or perceived figures. Like arithmetic, which was prized by the Greeks as the purer science, geometry should be a science of abstractions based on logical rules. But hypothetical spatial or kinetic constructions were required for geometrical proofs, and these concepts entered mathematics as postulates, quasi-empirical assumptions not necessarily accepted by a hypothetical interlocutor.

Plato pointed out that many missed the point of postulation as a means of identifying the anti-empirical character of mathematical 'points', 'lines', and other figures. A 'line' was not a real line, a 'point' was not a point, and the evidence used to prove geometrical theorems was in no way sensual or dependent on the world of becoming. On the other hand, geometers had nothing like the logically complete theory of Being that was developed by the Eleatics, or Pythagorean arithmetic, and the axiomatic framework for geometry in Euclid may have been expressly motivated by a need to accommodate geometrical ideas in the new mathematics; i.e. axioms were *needed* by geometers because their proofs were suspect by contemporary standards of rigor. While the arithmetic of numbers was rid of pebble-proofs in favor of deduction and indirect proofs, geometers had a more difficult time in making the transition to logical practice. The aim of eliminating the pointing of *deiknymi*, or any other empirical act, from mathematical practice was only partially achieved in geometry, because of the complex network of postulates and axioms. While arithmetic may have been seen as a reasonable extension of Eleatic philosophy, geometers' use of deductive proof was somewhat *ad hoc*, and required the novel technique of postulation and axiomatization. Regardless of the many technical errors in Euclid's geometry, it was not apparent whether the rules for the language-games of geometrical theory were the same as those for arithmetic. Plato's complaint that

> geometry is in direct contradiction with the language employed in it by its adepts . . . [that] they speak as if they were doing something [practical] . . . [that] their talk is of squaring and applying and adding and the like, whereas in fact the real object of the study is pure knowledge,

reflected the problems geometers had in characterizing what was intended by 'the square as such and the diagonal as such, and not for the sake of the image of it which they draw . . .'.[26] While geometers should deal in 'lines', not lines, in 'points', not points, and the 'diagonal', not the diagonal, the technical achievement of that goal depended on a marriage of methods from poetry – for indirect proof – and dialectic – for proof structure.

Plato says in the *Timaeus* that geometry employs a kind of 'spurious' or 'bastard' reasoning, used to understand a 'third world' of space that lay between the ordinary empirical world and the world of mathematical form.[27] This was Plato's means of justifying geometrical truth while recognizing that it lacked the complete rigor of arithmetic. Space, like Being, was indestructible, but was a 'container' for the world of becoming and passing away. By Plato's time, Parmenides' distinction between knowledge and ignorance required the resurrection of 'opinion' (which Parmenides had dropped to facilitate his binary logic) as a third kind of knowledge to accommodate geometry as part of theoretical mathematics. Unlike arithmetic, geometry had no indivisible One, no simple starting-point, and required infinitely divisible lines – all of which were contradicted by Eleatic philosophy, though not exactly by Eleatic method. Axiomatic reasoning allowed geometers to proceed from arguably 'false' assumptions. Not until Aristotle did this purely dialectical solution to geometers' methodological problems become – by fiat – an absolute foundation for mathematical truth. None the less, geometrical mathematics in the time of Plato was impure; it was a bastard science of space.

VII

We are now finally in a position to answer the question posed at the start of this essay, whether some mathematics is intrinsically literate or not. I have suggested already why general features of oral traditions are antithetical to mathematical proof. I shall conclude with a sketch of how the historical and anthropological issues supporting this claim arise in a particular set of contemporary mathematical theorems. But first a reprise of the points covered so far.

The economy, true analysis, and non-empiricism of the alphabet makes possible in an extended way three essential properties of mathematical discourse: context-freedom, autonomy, and referentiality. These properties are in direct opposition to the context dependency of speech and the immersion of the performing subject that is characteristic of bardic-Homeric discourse. The 'concepts' of the latter are characterized by paratactic aggregates and action-oriented descriptions, while the alphabet aids in the systematic development of timeless abstractions such as Being, or a mathematical theory of One. By integrating the timeless syntax of such notions with Eleatic logic and the dialectical methods of offering axioms and postulates, mathematicians were able to develop the idea of a deductive proof for theoretical assertions having absolutely no empirical basis. Hence the attack, begun by Parmenides and other pre-Socratics in poetry, against the world of becoming and the evidence of the senses, became allied with other strategies from oral argument to exclude empiricism, pointing, and pebble-proofs from mathematics – indeed to exclude any vestige of the Homeric world-view. A high point of this development was the proof of the incommensurability of the diagonal of a square with its side: the intrinsically new, and abstract notion of an incommensurable magnitude shows, at least in hindsight, that the conceptual difference introduced via the alphabet – that of an abstract consonant – could be not just the basis for full literacy, but could itself define a new category of thinking in its own right. For mathematicians, however, the imperfection in this massive transformation of ideas was the problem of eradicating empiricism completely from geometry while still reaping the benefits of Parmenidean proof techniques. *The sins of the fathers are visited upon the sons.*

The answer to the question 'How literate is mathematics?', lay in one of the central methods of modern logic mentioned above, that of Gödel-numbering or mathematical quotation. Remarkably, Kurt Gödel's famous theorems on mathematical incompleteness and the unprovability of consistency recapitulate three of the major conceptual developments of the Greeks sketched out in this essay: the development of the alphabet is repeated as the means *par excellence* of metamathematical abstraction; the use of the diagonal is repeated as a means of generating an intrinsically new mathematical category; and finally the problems faced by Greek geometers in eradicating empiricist methods from their proof procedures is repeated in a troublesome aspect of one of Gödel's theorems. That each one of these conceptual breakthroughs or problems of the Greeks has an analogue playing a fundamental role in the development of metamathematical theory may ultimately be of greater mathematical importance than Gödel's skeptical conclusions for the possibility of mathematical foundations. The presence of these elements in Gödel's proofs indirectly reflects the intrinsically literate character of mathematical proof. We may say that modern mathematical logic has finally shown that mathematical proof cannot be part of a traditional oral world and that this is part of what Gödel's proofs prove.

What did Gödel do? In 1931 he proved two startling theorems about the self-limitations of mathematical theory. The first theorem showed that mathematics was inherently incomplete in the sense that any tentative set of axioms (in the usual formal sense) left *some* mathematical assertion either unproved *or* unrefuted: such a sentence was *undecidable* with respect to the system of axioms at hand. The second theorem proved by Gödel was also negative, antifoundational, and skeptically motivated. Gödel showed that *if* mathematics as a formal system was consistent *then* you could not prove that fact mathematically without circularity: the self-consistency of mathematics was unprovable. These two theorems have had a decisive impact on the development of mathematics, logic, and the science of computers over the last half-century, not the least because Gödel took the relatively vague notions of proof, theorem, consistency, etc. and recast them in a theory of mathematically rigorous concepts. Gödel proved new results by devising a way of looking at logical and metamathematical problems so that they could be given a systematic mathematical analysis; his methodology was revolutionary and *sui generis*.

In the hindsight of this essay, one of Gödel's discoveries was that the conditions of context-freedom, autonomy, and referentiality – of axioms, formulas, theorems, and proofs – could all be expressed as formal mathematical conditions. Gödel was the first mathematician and philosopher after Gottlob Frege systematically to extend *mathematical* standards of mathematical objectivity that were largely consistent with mathematical practice. Efforts by mathematicians such as David Hilbert and L. E. J. Brouwer failed to develop all three conditions of mathematical discourse into mathematical, rather than psychologistic or metaphysical, standards. Gödel makes all three conditions mathematically precise. At the same time, Gödel was the only mathematician of his day even to consider the notion of undecidability or the unprovability of consistency, both of which were wholly antithetical to the positivist framework in which others worked. It was Gödel who made metamathematics *into* a branch of mathematics instead of technical philosophy.

So what can be said of Gödel's novel methodology and its prefiguration in ancient history? First, Gödel's principle for coding theorems and proofs as numbers through a system of structured names $w^i x^j y^k$ is a mathematical generalization of the analytic principle underlying the alphabet. Instead of a relatively limited scheme of vowels and consonants, Gödel-numbering combines all the logical operators, categories, and relations needed to define the syntax of mathematical statements. A concrete example of a sequence of Gödel-numbers is the series of numbers represented by all the patterns of zeros and ones internally stored in a computer as it performs a calculation, such as adding a column of numbers, or searching for a string of text in a word processor. Though historically the situation may be reversed, Gödel showed the alphabetic principle to be at bottom mathematical, and that the technique could be used to represent mathematics *as* mathematical theory – as arithmetic theory to be precise, since Gödel's codes reduce completely to additions and multiplications.

But if Gödel's number-coding is essentially a mathematically rigorous development of alphabetic analysis, how could his proof make sense – other limitations notwithstanding – to an oral culture like the Homerics', whose entire world-view is shaped by discursive practices which oppose and resist the context-freedom, autonomy, and referentiality made possible by alphabetic literacy? It would seem that for Gödel's proofs to be proved by and for genuine non-literates, they would have to grasp some essential principles of literate discourse and in effect deny fundamental principles of their own discursive practices. There is no alphabetic writing in the abstract that might be deployed independently of a form of life – the alphabet helps make much of abstraction possible, and it utilizes consonants *as* a key abstraction, but the alphabet itself is a contingent historical development in human communication. To be able to conceive of Gödel-numbering in an oral or preliterate culture would be a linguistic and cultural contradiction. On the other hand, we do not question Gödel's alphabetization because, as members of a literate tribe, the conditions of literate mathematical discourse are – almost – transparent to the user. Gödel's proofs are not pure mathematics, they are interpreted mathematics, like the geometry of the Greeks, and even calculus around 1800 which allowed physicalist arguments on heat transfer and vibrating strings to be part of mathematical proofs. *The interpretative metaphor here is language itself*, derived from conditions specific to written, alphabetic language only.[28]

A second fundamental idea used by Gödel, which exploited a technique that had appeared earlier in the mathematics of Bertrand Russell and Georg Cantor, is that of diagonalization. Gödel's famous self-referential, undecidable sentences, which say of themselves, 'I am not provable', are constructed by 'crossing', in a manner made perspicuous through a diagrammatic (doubly infinite) square and diagonal, what a mathematical sentence *says* with the mathematical object which it *describes*.[29] The trick of diagonalization is to show one can create self-denoting and self-describing sentences; the *application* is then to use such properties systematically to create mathematical sentences whose truth falls *outside* that represented in a pre-specified axiom system. Diagonalization formally generates categorical differences, similar to the difference between rational and irrational quantities, between whole classes of mathematical concepts. In fact, diagonalization arguments are today found throughout the branches of mathematical logic and are almost always used with this purpose in mind. Gödel finally made the creation of the 'new' a positive mathematical heuristic that is now a well-used tool of all logicians; the basis of this method is the simple system of alphabetic additions and multiplications defining Gödel-numbering as mathematical quotation.

On the one hand then, Gödel's work represents a triumph of representing mathematics as literate language. Yet what happens in Gödel's proofs is that his powerful referential calculus leads to an explicit inter-embedding of linguistic modalities. At one crucial point in the proof of Gödel's Second Incompleteness Theorem, *informal* mathematical proofs must be treated as inspectable things, or quasi-empirical linguistic constructs which can themselves be referred to and described in establishing further results.[30]

Gödel's theorems violate the Platonic admonition about confusing mathematical 'lines' with lines and mathematical 'squaring' with squaring; indeed this transgression of abstract and empirical boundaries is one of the most curious and exquisite features of Gödel's work. To prove the Second Incompleteness Theorem requires two steps. First, one proves the First Theorem on undecidability; *then* one derives the Second Theorem on the unprovability of consistency as a simple corollary to the First. But in this new proof, which is only a few lines long, the essential idea is, quite literally, to inspect – really to reflect upon, as one reflects upon a pebble-proof – the prior proof, which develops the theory of Gödel-numbering and the necessary diagonalization constructions, to see that the proof meets certain formal conditions. From this pointing out, and some simple arguments, one derives the Second Theorem. The perplexing notion here is the referent of 'the proof being inspected', and whose characteristics are pointed out. Is it a legitimate mathematical entity, a Platonic 'proof', or is it just a proof? Apparently it is the latter, since the whole point of the argument is to claim, by inspection, that the proof of the First Theorem *can* be translated into a desired *formal* structure. The Second Theorem is, indeed, an informal theorem *about* the translatability of another informal theorem into formal logic. In so far as this part of the proof is essential, the proof lies in a third world between the concrete examples of logical calculi found in computer programs, and the world of true, virtual, mathematical abstraction. But it is precisely the latter that is being characterized through metamathematical representation! We are led to conclude that the only consistent criterion that can be applied is that mathematical standards are literate standards, and that literacy is an essential condition for mathematical theory as we know it. If one accepts the broad paradigm laid down by Gödel for metamathematical analysis (which accounts for much), then it not only follows that mathematics is literate mathematics, but that to complete Gödel's logical theory in the unprovability of consistency, one has to point out certain fragments of this literate practice in making a logical-mathematical proof; so if the proof is not *there*, you cannot complete the proof.

All these conclusions appear paradoxical if one believes in genuine Platonism. But virtual Platonism is all we need for mathematics, and hence mathematics, in addition to enjoying the advantages of the written languages and discourse which make virtual Platonism possible, also suffers their slight imperfections. If, as Gödel has shown, mathematics *is* written language, then the contingencies of written language belong to mathematics as well. Logic is an applied science, and, at least at times, Gödel is talking of proof, not 'proof', theorems, not 'theorems'. Were Plato to inspect Gödel's Second Theorem he might conclude that 'logic is a bastard science', or at least that Gödel's proof should end with *deiknymi* instead of 'QED'. The parallel between the quasi-empirical geometry of Plato's day and logic in our own – as an interpreted calculus on its way to pure mathematics – is a useful one, and may provide guidance to mathematical logic itself. Which is all that I wanted to show.

NOTES

1 Don DeLillo, *Ratner's Star* (New York: Random House; Vintage Edition, 1980), p. 337.
2 Noted in Paul Feyerabend, *Farewell to Reason* (London: New Left Books, 1987), p. 111. This book contains a wealth of insights on the relationship between Homeric, pre-Socratic, and Platonic epistemologies.
3 Andrew Hodges, *Alan Turing: The Enigma* (New York: Simon & Schuster, 1983), p. 530.
4 Aspects of linguistic objectivity are discussed by Paul Feyerabend in *Problems of Empiricism: Philosophical Papers II* (Cambridge: Cambridge University Press, 1981), chs 1, 9. Feyerabend criticizes and extends Karl Popper's notions developed in his *Objective Knowledge: An Evolutionary Approach* (Oxford: Oxford University Press, 1975).
5 My discussion of Greek mathematics draws fundamentally on Árpád Szabó, *The Beginnings of Greek Mathematics*, trans. by A. M. Ungar (Dordrecht: D. Reidel, 1978).
6 See, e.g., *Parmenides* 128b, in Edith Hamilton and Huntington Cairns (eds), *Plato: The Collected Dialogues* (Princeton: Bollingen, 1963): '[T]heir own supposition that there is a plurality leads to even more absurd consequences than the hypothesis of the one.' Before Aristotle, contradiction is a relative not absolute relation for the Greeks, a view that has been rediscovered today through the work of Kurt Gödel.
7 'Parmenidean syntax insists that there be no subject, for it is asserting the naked proposition that any and every statement describing the environment or any part or aspect of it should be an "is" statement cast, as we might say, *in the form of a theorem*'; Eric Havelock, 'The Linguistic task of the Presocratics', in Kevin Robb (ed.), *Language and Thought in Early Greek Philosophy* (LaSalle: The Hegeler Institute, 1983), p. 26 (my emphasis).
8 *Parmenides* 128b.
9 See *Beginnings of Greek Mathematics*, ch. 3. More precisely, proofs end with 'being what it was required to prove'.
10 See Thomas Heath, *A History of Greek Mathematics I* (New York: Dover Editions, 1981 reprint of 1921), p. 373.
11 On the alphabet, see I. J. Gelb, *A Study of Writing* (2nd ed.) (Chicago: University of Chicago, 1963). Gelb emphasizes the development of Greek vowel structure as central to the completion of the alphabet, while Eric Havelock has emphasized how this change was reflected in the abstract role then explicitly given to consonants. Most of what I say about the role of literacy and orality in Greek philosophy and culture comes from Havelock, especially: *The Literate Revolution in Greece and its Cultural Consequences* (Princeton: Princeton University Press, 1982), chs 3, 4; *Preface to Plato* (Harvard: Harvard University Press, 1963); 'The Linguistic Task of the Presocratics', op. cit.
12 *Theaetetus* 203b, my emphasis.
13 See *Literate Revolution*, op. cit., p. 81.
14 On the problems of syllabaries, see John Chadwick, *The Decipherment of Linear B* (2nd ed.) (Cambridge: Cambridge University Press, 1967), p. 22. See also, e.g., p. 32 on the role of determinatives as a means for classifying word types.
15 See *Preface to Plato*, op. cit., ch. 4.
16 See B. L. van der Waerden, *Science Awakening: Egyptian, Babylonian, and Greek Mathematics* (trans. by A. Dresden) (New York: John Wiley & Sons, 1963), p. 54.
17 See Albert Lord, *The Singer of Tales* (Harvard: Harvard University Press, 1960), pp. 25, 99 ff.
18 See *Science Awakening*, op. cit., p. 266 on 'the difficulties of the written tradition' in Greek mathematics, accountable to the oral traditions maintaining the meaning of the proofs. 'As long as there was no interruption, as long as each generation could hand over its method to the next, everything went well', but if there was 'an interruption in the oral tradition, and only books remained, it became extremely difficult to assimilate the work of the great precursors and next to impossible to pass beyond it'. My argument in the text is that in oral traditions this practical limitation becomes a theoretical limitation on the possibility of mathematical proof.

19 See *Preface to Plato*, op. cit., ch. 2.
20 *Republic* 393.
21 *Republic* 605d.
22 See *Literate Revolution*, op. cit., p. 17.
23 Bruno Snell, *The Discovery of the Mind* (trans. by T. G. Rosenmeyer) (New York: Dover Editions, 1982), ch. 1. On the role of parataxis in Homeric thought, see also Paul Feyerabend, *Against Method* (London: New Left Books, 1975), ch. 17.
24 On axioms and hypotheses, see *Beginnings of Greek Mathematics*, op. cit., ch. 3. The perspective of ancient axiomatics is today commonplace in modern set theory (see note 6).
25 See *Science Awakening*, op. cit., p. 108.
26 *Republic* 527, 518d. See *Science Awakening*, op. cit., p. 150, for an example of a proof demonstrating 'Archytas' kinematical way of thinking [for which a point . . .] is not a definite point in space, but a moving point'. The proof in question is for the famous 'Delian problem', or the duplication of the cube.
27 'Bastard' is Szabó's translation; see *Beginnings of Greek Mathematics*, op. cit., p. 311.
28 See *Science Awakening*, op. cit., p. 37 on the importance of good notation as necessary for mathematical progress, here with respect to the Babylonians' positional notation, which is like our use of the decimal system. Again (see note 18) my argument is that for certain parts of logic, this practical advantage is a theoretical necessity.
29 In 'Philosophy and the Written Word' (in Robb, op. cit., see note 7), pp. 117 f., Charles Kahn notes of one of Heraclitus' fragments that 'we must read that sentence *twice* in order to see that the adverb *aiei* "always" may be construed either with what precedes ("this *logos is so*") or with what follows ("men fail to understand"). Now it is one of the chief advantages of reading to oneself that it is easier to go back and start over. And this will be necessary for the comprehension of many of the fragments'. Gödel's undecidable sentence similarly is read twice: first to understand that it says that *some* sentence is unprovable, and then a second time to see that the unprovable sentence *is* that very sentence. Kahn also says of Heraclitus' fragments that 'as an oral text, spoken but not repeated, it would be sheer mystification . . . Heraclitus is the first philosopher for whom the *written word* is the essential form of communication'. So, too, for Gödel as mathematician.
30 For a detailed discussion, see my 'Reflections on the Legacy of Kurt Gödel', *Philosophical Forum 20*, Spring 1989.

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